Conformal blocks of C_2-cofinite vertex operator algebras

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Vertex operator algebras

- A vertex operator algebra(VOA) is a graded vector space $\mathbb{V}$ together with a vertex operator

$$\mathbb{V} \to \mathrm{End}(\mathbb{V})[[z^{\pm 1}]], \quad v \mapsto Y(v,z) = \sum_{n\in\mathbb{Z}} Y(v)_n z^{-n-1}$$

  satisfying certain conditions.
- $\mathbb{V}$ is called C_2-**cofinite** if $\mathrm{Span}\{Y(u)_{-2}v : u, v \in \mathbb{V}\}$ has finite codimension. If $\mathbb{V}$ is C_2-cofinite, then $\mathbb{V}$ has finitely many irreducibles, fusion rules are finite, etc.
- $\mathbb{V}$ is called **rational** if all "admissible" $\mathbb{V}$-modules are semisimple.

$\mathrm{Rep}(\mathbb{V})$ as a modular tensor category

- From 1995 to 2008, Huang and Lepowsky proved that $\mathrm{Rep}(\mathbb{V})$ together with the fusion product $\boxtimes$ is a rigid modular tensor category when (the CFT type and self-dual) $\mathbb{V}$ is C_2-cofinite and rational.
- When $\mathbb{V}$ is only C_2-cofinite, Huang-Lepowsky-Zhang constructed a braided tensor category for the representation of $\mathbb{V}$. However, proving rigidity and modularity is still an open problem. This is partly because genus 1 conformal block theory of C_2-cofinite VOAs is not clear.

What is a conformal block?

A **conformal block** associated to a pointed surface $\mathfrak{X}$ (a compact Riemann surface C with N marked points $x_1, \cdots, x_N$) is a linear map $\varphi : \mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N \to \mathbb{C}$ invariant under the "action" of $\mathbb{V}$. Here $\mathbb{W}_1, \cdots, \mathbb{W}_N$ are associated to $x_1, \cdots, x_N$. The space of conformal blocks associated to $\mathfrak{X}$ and $\mathbb{W}_1, \cdots, \mathbb{W}_N$ is denoted by $\mathscr{T}^*_{\mathfrak{X}}(\mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N)$.

Examples of genus 0 conformal blocks

Assume that $\mathbb{V}$ is C_2-cofinite and rational. Let $\mathcal{E}$ be the set of all equivalence classes of irreducible $\mathbb{V}$-modules.

- Let $\mathbb{W}_1, \mathbb{W}_2$ be semisimple $\mathbb{V}$-modules and $\mathfrak{P} = (\mathbb{P}^1; 0, \infty)$ be a pointed surface. Associate $\mathbb{W}_1, \mathbb{W}_2^\vee$ to $0, \infty$. Then

$$\mathscr{T}_{\mathfrak{P}}^*(\mathbb{W}_1 \otimes \mathbb{W}_2^\vee) \simeq \mathrm{Hom}_{\mathbb{V}}(\mathbb{W}_1, \mathbb{W}_2).$$

- Let $\mathbb{W}_1, \mathbb{W}_2, \mathbb{W}_3$ be semisimple $\mathbb{V}$-modules and $\mathfrak{P}_\xi = (\mathbb{P}^1; 0, \xi, \infty)$ be a pointed surface. Associate $\mathbb{W}_2, \mathbb{W}_1, \mathbb{W}_3^\vee$ to $0, 1, \infty$. Then

$$\mathscr{T}_{\mathfrak{P}_\xi}^*(\mathbb{W}_2 \otimes \mathbb{W}_1 \otimes \mathbb{W}_3^\vee) \simeq \mathrm{Hom}_{\mathbb{V}}(\mathbb{W}_1 \boxtimes \mathbb{W}_2, \mathbb{W}_3).$$

Fusion products in $\mathrm{Rep}(\mathbb{V})$

- More precisely, the fusion product in $\mathrm{Rep}(\mathbb{V})$ can be described by

$$\mathbb{W}_1 \boxtimes \mathbb{W}_2 = \bigoplus_{\mathbb{M} \in \mathcal{E}} \mathbb{M} \otimes \mathscr{T}_{\mathfrak{P}_\xi}(\mathbb{W}_2 \otimes \mathbb{W}_1 \otimes \mathbb{W}_3^\vee).$$

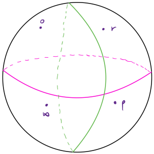
- Since $\mathbb{V}$ is C_2-cofinite, the space of conformal blocks is finite dimensional. This means that fusion rules in $\mathrm{Rep}(\mathbb{V})$ are all finite.
- This implies, among other things, that fusion products and fusion rules are characterized by genus 0 conformal blocks.

Associativity in $\mathrm{Rep}(\mathbb{V})$

The associativity isomorphism

$$\mathcal{A} : (\mathbb{W}_1 \boxtimes \mathbb{W}_2) \boxtimes \mathbb{W}_3 \xrightarrow{\simeq} \mathbb{W}_1 \boxtimes (\mathbb{W}_2 \boxtimes \mathbb{W}_3).$$

is determined by conformal block factorizations of two types of factorizations of the pointed surface $\mathfrak{P}^2_{r,\rho} = (\mathbb{P}^1; 0, r, \rho, \infty)$:

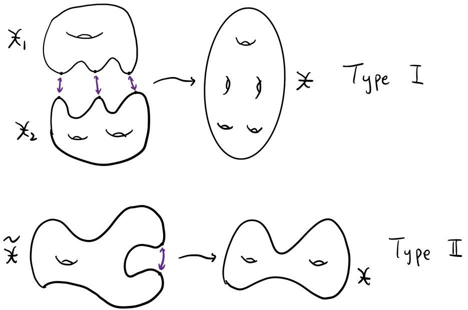


where $0 < r < \rho < \infty$.

The associativity in $\mathrm{Rep}(\mathbb{V})$ is characterized by sewing and factorization of genus 0 conformal blocks.

Sewing compact Riemann surfaces

We can sew (pointed) compact Riemann surfaces $\widetilde{\mathfrak{X}}$ to get (pointed) compact Riemann surfaces $\mathfrak{X}$ of higher genus.

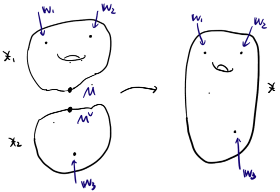


Sewing conformal blocks

Sewing conformal blocks is defined by taking contractions. By sewing conformal blocks associated to $\widetilde{\mathfrak{X}}$, we obtain new conformal blocks associated to $\mathfrak{X}$.

Type I: If we choose a conformal block φ_1 (resp. φ_2) associated to $\mathfrak{X}_1$ (resp. $\mathfrak{X}_2$), then we can sew φ_1 and φ_2 to obtain

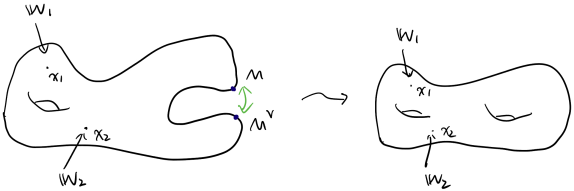
$$\varphi(w_1 \otimes w_2 \otimes w_3) := \varphi_1(w_1 \otimes w_2 \otimes -)\varphi_2(w_3 \otimes -)$$



Sewing conformal blocks

Type II: If we choose a conformal block $\widetilde{\varphi}$ associated to $\widetilde{\mathfrak{X}}$, then we can sew $\widetilde{\varphi}$ to obtain

$$\varphi(w_1 \otimes w_2) := \widetilde{\varphi}(w_1 \otimes w_2 \otimes \overbracket{- \otimes -})$$



Sewing and factorization theorem, rational case

Assume that the pointed surface $\mathfrak{X}$ is obtained by sewing a single pair of points of the pointed surface $\widetilde{\mathfrak{X}} = \mathfrak{X}_1 \sqcup \mathfrak{X}_2$ for simplicity. Then sewing conformal blocks gives a linear map

$$\mathcal{S} : \bigoplus_{\mathbb{M}\in\mathcal{E}} \mathscr{T}^*_{\mathfrak{X}_1}(\mathbb{W}_1 \otimes \mathbb{W}_2 \otimes \mathbb{M}) \otimes \mathscr{T}^*_{\mathfrak{X}_2}(\mathbb{W}_3 \otimes \mathbb{M}') \to \mathscr{T}^*_{\mathfrak{X}}(\mathbb{W}_1 \otimes \mathbb{W}_2 \otimes \mathbb{W}_3)$$

$\mathcal{S}$ is well-defined and injective due to Gui ('20).
Damiolini-Gibney-Tarasca ('19) proved that

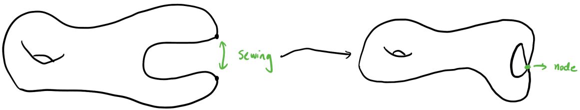
$$\sum_{\mathbb{M}\in\mathcal{E}} \dim \mathscr{T}^*_{\mathfrak{X}_1}(\mathbb{W}_1 \otimes \mathbb{W}_2 \otimes \mathbb{M}) \dim \mathscr{T}^*_{\mathfrak{X}_2}(\mathbb{W}_3 \otimes \mathbb{M}') = \dim \mathscr{T}^*_{\mathfrak{X}}(\mathbb{W}_\bullet)$$

This proves that $\mathcal{S}$ is an isomorphism.

Key point of DGT's proof

DGT's proof uses the **level 0 Zhu's algebra** $A(\mathbb{V})$. The representations of $A(\mathbb{V})$ coorespond to highest weight representations of $\mathbb{V}$. So it can determine the whole representations when $\mathbb{V}$ is rational.

The geometric interpretation of $A(\mathbb{V})$ is **nodal curves**. They reduce the general case to "nodal factorization", which is easier to deal with.



Folklore: in the rational case, the level 0 Zhu's algebra and nodal curves determine the whole story.

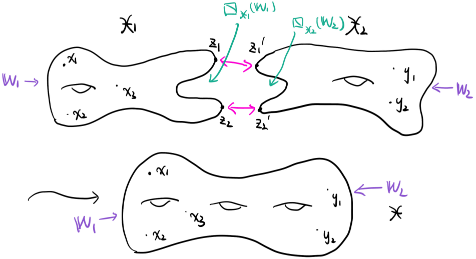
Sewing and factorization, nonrational case

Now *we assume that* $\mathbb{V}$ *is only* C_2*-cofinite and not necessarily rational*. We emphasize that:

- It is not enough to associate $\mathbb{V}$-modules $\mathbb{W}_1, \cdots, \mathbb{W}_N$ to the marked points separatedly. Instead, we associate $\mathbb{V}^{\otimes N}$**-module** $\mathbb{W}$ to the marked points. In general, a $\mathbb{V}^{\otimes N}$ module cannot be written as a direct sum of tensor products of $\mathbb{V}$-modules.
- It is not enough to consider the self-sewing case, i.e., type II sewing. Moreover, it is not complete to consider type I sewing along single pair of points.

Sewing and factorization, nonrational case

We only need to consider the disjoint sewing of the following type:

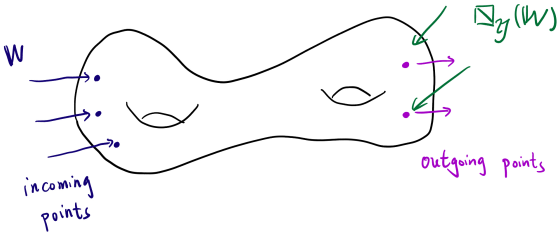


Sewing and factorization, nonrational case

- Modular invariance for C_2-cofinite and rational VOAs is related to character $\mathrm{Ch}(\mathbb{M}) = \sum_{n=0}^{\infty} \dim(\mathbb{M}_n) q^{n-\frac{c}{24}}$. The geometric aspect of this character is self-sewing, due to Zhu ('96).
- The above proof of modular invariance does not work when $\mathbb{V}$ is only C_2-cofinite. This fact was first realized by Miyamoto ('04). He introduced a new kind of trace map, called **pseudotrace**, and proved that the space spanned by pseudotrace satisfies the modular invariance property. The geometric aspect of pseudotrace is difficult to understand. This is one of our goals in the program.

Dual fusion products

Suppose $\mathfrak{Y} = (y_1, \cdots, y_M | C | x_1, \cdots, x_N)$ is a pointed surface. $x_1, \cdots, x_N$ are incoming points and $y_1, \cdots, y_M$ are outgoing points. We call $\mathfrak{Y}$ an (M, N)-**pointed surface** for simplicity.



Dual fusion products

Definition

Associate a $\mathbb{V}^{\otimes N}$-module $\mathbb{W}$ to $x_1, \cdots, x_N$. A **dual fusion product associated to** $\mathbb{W}$ **and** $\mathfrak{Y}$ is a $\mathbb{V}^{\otimes M}$-module $\boxbslash_{\mathfrak{Y}}(\mathbb{W})$ contained in $\mathbb{W}^*$ such that: if we associate $\boxbslash_{\mathfrak{Y}}(\mathbb{W})$ to $y_1, \cdots, y_M$, then

(1) the natural pairing $\omega : \mathbb{W} \otimes \boxbslash_{\mathfrak{Y}}(\mathbb{W}) \to \mathbb{C}$ is a conformal block in $\mathscr{T}_{\mathfrak{Y}}^*\big(\mathbb{W} \otimes \boxbslash_{\mathfrak{Y}}(\mathbb{W})\big)$. It is called a **canonical conformal block**.

(2) for any $\mathbb{V}^{\otimes M}$ module $\mathbb{M}$ and any conformal block $\Psi : \mathbb{W} \otimes \mathbb{M} \to \mathbb{C}$ associated to $\mathfrak{Y}$, there exists a unique homomorphism $\Phi : \mathbb{M} \to \boxbslash_{\mathfrak{Y}}(\mathbb{W})$ such that $\Psi = \omega \circ (\mathbf{1} \otimes \Phi)$.

The contragredient module $\boxtimes_{\mathfrak{Y}}(\mathbb{W})$ of $\boxbslash_{\mathfrak{Y}}(\mathbb{W})$ is called a **fusion product associated to** $\mathfrak{Y}$ **and** $\mathbb{W}$.

Dual fusion products, universal property

Theorem (Gui-Z. '23)

Assume $\mathbb{V}$ is C_2-cofinite. Associate a $\mathbb{V}^{\otimes N}$-module $\mathbb{W}$ to $x_1, \cdots, x_N$. There exists a unique dual fusion product $\boxbslash_{\mathfrak{Y}}(\mathbb{W})$ associated to $\mathbb{W}$ and $\mathfrak{Y}$.

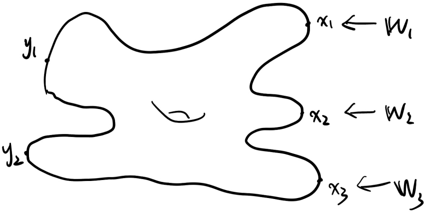
The main difficulty in this theorem is the module structure of $\boxbslash_{\mathfrak{Y}}(\mathbb{V})$. In the rational case, Zhu, Damiolini-Gibney-Tarasca defined the module structure using level 0 Zhu's algebras. This method does not work in the nonrational case. We introduce **propagation of dual fusion products** to define the module structure.

Examples of (dual) fusion products

- If we choose $\mathfrak{Y} = (0, \infty \big| \mathbb{P}^1 \big| 1)$ and associate $\mathbb{V}$ to 1, then $\boxbslash_{\mathfrak{Y}}(\mathbb{V})$ covers (higher level) Zhu's algebras, due to Zhu ('96) and Dong-Li-Mason ('98).
- Assume $\mathbb{V}$ is C_2-cofinite and rational. If we choose $\mathfrak{Y} = (\infty \big| \mathbb{P}^1 \big| 0, 1)$ and associate $\mathbb{W}_1, \mathbb{W}_2$ to $0, 1$, then $\boxtimes_{\mathfrak{Y}}(\mathbb{W}_1 \otimes \mathbb{W}_2)$ is the usual fusion product in $\mathrm{Rep}(\mathbb{V})$.

Examples of (dual) fusion products

- Assume $\mathbb{V}$ is C_2-cofinite and rational. If



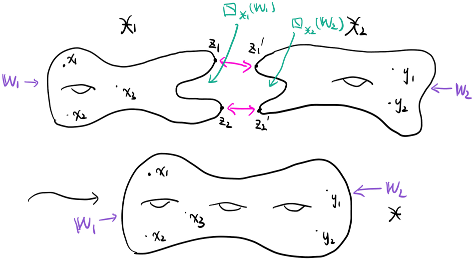
then

$$\boxtimes_{\mathfrak{Y}}(\mathbb{W}_1 \otimes \mathbb{W}_2 \otimes \mathbb{W}_3)$$

$$\simeq \bigoplus_{\mathbb{M}_1,\mathbb{M}_2 \in \mathcal{E}} \mathbb{M}_1 \otimes \mathbb{M}_2 \otimes \mathscr{T}^*_{\mathfrak{Y}}(\mathbb{W}_1 \otimes \mathbb{W}_2 \otimes \mathbb{W}_3 \otimes \mathbb{M}_1 \otimes \mathbb{M}_2)$$

Sewing and factorization, nonrational case

Assume the following setting



The canonical conformal block on $\mathfrak{X}_1$ (resp. $\mathfrak{X}_2$) is denoted by ω_1 (resp. ω_2).

Factorization of conformal blocks, nonrational case

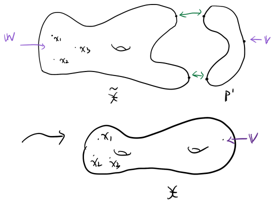
Theorem (Gui-Z. '23)

The linear map

$$\begin{gathered}\mathrm{Hom}_{\mathbb{V}^{\otimes 2}}\big(\boxtimes_{\mathfrak{X}_2}(\mathbb{W}_2), \boxbslash_{\mathfrak{X}_1}(\mathbb{W}_1)\big) \to \mathscr{T}_{\mathfrak{X}}^*(\mathbb{W}_1 \otimes \mathbb{W}_2)\\ T \mapsto \sum_i \omega_1(w_1 \otimes T e_i^\vee)\omega_2(w_2 \otimes e_i)\end{gathered}$$

is an isomorphism. Here $\{e_i\}$ *is a basis of* $\boxbslash_{\mathfrak{X}_2}(\mathbb{W}_2)$ *and* $\{e_i^\vee\}$ *is a dual basis of* $\boxtimes_{\mathfrak{X}_2}(\mathbb{W}_2)$*.*

- Suppose $\mathbb{V}$ is C_2-cofinite. Consider



Then

$$\mathscr{T}^*_{\mathfrak{X}}(\mathbb{W}) \simeq \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}\Big(\boxtimes_{\mathbb{P}^1}(\mathbb{V}), \boxbslash_{\widetilde{\mathfrak{X}}}(\mathbb{W})\Big)$$

$\boxtimes_{\mathbb{P}^1}(\mathbb{V})$ is crucial and is closely related to (higher level) Zhu's algebras and Longo-Rehren subfactors.

Sewing and factorization, nonrational case

- If $\mathbb{V}$ is C_2-cofinite and rational, then

$$\boxtimes_{\mathbb{P}^1}(\mathbb{V}) \simeq \bigoplus_{\mathbb{M}\in\mathcal{E}} \mathbb{M}\otimes\mathbb{M}^\vee.$$

So

$$\mathscr{T}^*_{\mathfrak{X}}(\mathbb{W}) \simeq \bigoplus_{\mathbb{M}\in\mathcal{E}} \mathscr{T}^*_{\widetilde{\mathfrak{X}}}(\mathbb{W}\otimes\mathbb{M}\otimes\mathbb{M}^\vee)$$

This covers sewing and factorization theorem given by Gui and Damiolini-Gibney-Tarasca in self-sewing case.