How are (co)ends related to pseudo-traces?

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Modular invariance

- A central theme in vertex operator algebras (VOAs) is modular invariance.
- An early breakthrough in this topic is Zhu's theorem (96): Assume that $\mathbb{V}$ is "***C_2*-cofinite** and rational". Then the set of all $\mathrm{Tr}_{\mathbb{M}} q^{L_0 - \frac{c}{24}}$ span a $SL(2,\mathbb{Z})$-invariant space (where $\mathbb{M} \in \mathrm{Mod}(\mathbb{V})$).
- More generally: Let $Y(v,z)$ denote the vertex operators (where $v \in \mathbb{V}$). Then $(v,\tau) \in \mathbb{V} \times \mathbb{H} \mapsto \mathrm{Tr}_{\mathbb{M}} Y(v,1) q_\tau^{L_0 - \frac{c}{24}}$ (over all $\mathbb{M} \in \mathrm{Irr}$) span an $SL(2,\mathbb{Z})$-invariant space with dimension $\#\mathrm{Irr}(\mathbb{V})$. Here $q_\tau = e^{2\pi \mathbf{i} \tau}$.
- Note: "C_2-cofinite" is a finiteness condition ensuring, e.g., that $\mathrm{Mod}(\mathbb{V})$ is a finite abelian category. We always assume C_2-cofinite in the talk. "Rational" means that $\mathrm{Mod}(\mathbb{V})$ is semi-simple.

Modular invariance beyond rationality

- However, this modular invariance does not hold when rationality is dropped: $\mathrm{Tr}_{\mathbb{M}} q_\tau^{L_0 - c/24}$ is a fractional power of $q_\tau = e^{2\mathbf{i}\pi\tau}$. However, without rationality (e.g. triplet algebras $\mathcal{W}_p = \mathcal{W}_{1,p}$ and their generalizations $\mathcal{W}_{q,p}$, symplectic fermions $SF(d)^+$), an $SL(2,\mathbb{Z})$ action of $\mathrm{Tr}_{\mathbb{M}} q_\tau^{L_0 - c/24}$ will contain factors such as $\tau = \frac{1}{2\mathbf{i}\pi} \log q_\tau$.
- To rescue modular invariance, Miyamoto (04) introduced the **pseudo-q-trace** construction $(v,\tau) \in \mathbb{V} \times \mathbb{H} \mapsto \mathrm{Tr}^\omega_{\mathbb{M}} Y(v,1) q_\tau^{L_0 - \frac{c}{24}}$. For $\mathbb{M} \in \mathrm{Mod}(\mathbb{V})$, a pseudo-trace $\mathrm{Tr}^\omega_{\mathbb{M}}$ is a symmetric linear functional on a suitable subalgebra of $\mathrm{End}(\mathbb{M})$. This was later simplified by Arike-Nagatomo (11).
- Miyamoto showed that the pseudo-q-traces form an $SL(2,\mathbb{Z})$-invariant space whose dimension is characterized by the higher Zhu's algebras of $\mathbb{V}$.

Goal of the talk

- In the categorical and TFT approach, genus-1 data and modular invariance are understood in terms of ends and coends.

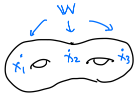
TQFT	ends and coends
VOA	pseudo-traces

  The goal of this talk is to explain how these two approaches are related under the framework of VOA conformal blocks and their sewing-factorization property.

- In the first part of the talk, I will explain how (co)ends naturally appear in VOA conformal blocks. In the second part, I will relate them to pseudo-traces.

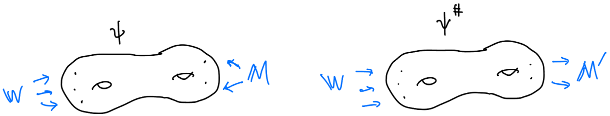
Conformal blocks (CB)

- Recall that $\mathbb{V}$ is always a C_2-cofinite $\mathbb{N}$-graded VOA.
- Fix a N-pointed compact Riemann surface with local coordinates $\mathfrak{X} = (C; x_1, \cdots, x_N; \eta_1, \cdots, \eta_N)$.
    - C is a (possibly disconnected) compact Riemann surface with distinct marked points $x_1, \ldots, x_N$. η_i is a local coordinate at x_i (i.e., an injective holomorphic function on a neighborhood of x_i sending x_i to 0).
- Associate $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$ to the *ordered* marked points $x_1, \cdots, x_N$.
- A **conformal block** (CB) is a linear map $\psi : \mathbb{W} \to \mathbb{C}$ invariant under the action defined by $\mathfrak{X}$ and $\mathbb{V}$ (Zhu 94, Frenkel&Ben-Zvi 04). The spaces of conformal blocks is denoted by $CB(\mathfrak{X}, \mathbb{W})$, or

$CB\Bigg($  $\Bigg)$

Pictorial illustration of CB

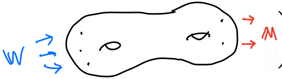
- Suppose that $\mathfrak{X}$ has two groups of marked points $x_1, \ldots, x_N$ and $y_1, \ldots, y_K$. We can associate $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$ to $x_1, \ldots, x_N$ in order, and associate $\mathbb{M} \in \mathrm{Mod}(\mathbb{V}^{\otimes K})$ to $y_1, \ldots, y_K$ in order.
  - This means associating $\mathbb{W} \otimes \mathbb{M}$ to $x_1, \ldots, x_N, y_1, \ldots, y_K$.
- A CB $\psi : \mathbb{W} \otimes \mathbb{M} \to \mathbb{C}$ can equivalently be viewed as a linear map $\psi^\sharp : \mathbb{W} \to \overline{\mathbb{M}'}$ satisfying certain intertwining property.



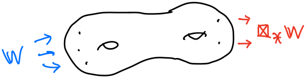
  - $\mathbb{M}'$ is the contragredient of $\mathbb{M}$, and $\overline{\mathbb{M}'}$ is the algebraic completion of $\mathbb{M}$. So $\overline{\mathbb{M}'} = \mathbb{M}^*$.

The fusion product $\boxtimes_{\mathfrak{X}}\mathbb{W}$ and the canonical CB $\gimel_{\mathfrak{X}}$

- The CB functor is left exact.
- Any left exact functor from a finite $\mathbb{C}$-linear category to $\mathcal{V}ect$ is representable (Douglas-SchommerPries-Snyder 19). So, fixing $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$, there exists $\boldsymbol{\boxtimes_{\mathfrak{X}}\mathbb{W}} \in \mathrm{Mod}(\mathbb{V}^{\otimes K})$, called the **fusion product** of $\mathbb{W}$ along $\mathfrak{X}$, yielding an equivalence of linear functors $\mathbb{M} \in \mathrm{Mod}(\mathbb{V}^{\otimes K}) \to \mathcal{V}ect$:

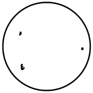
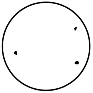
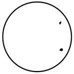
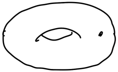
$$CB\Big($$ 

$$\Big) \simeq \mathrm{Hom}_{\mathbb{V}^{\otimes K}}(\boxtimes_{\mathfrak{X}}\mathbb{W}, \mathbb{M})$$

- The element $\gimel_{\mathfrak{X}} \in CB($ 

$)$ corresponding to $\mathrm{id} \in \mathrm{Hom}_{\mathbb{V}^{\otimes K}}(\boxtimes_{\mathfrak{X}}\mathbb{W}, \boxtimes_{\mathfrak{X}}\mathbb{W})$ is called the **canonical CB**.
    - $\gimel_{\mathfrak{X}}$ can be viewed as a linear map $\gimel_{\mathfrak{X}}^{\sharp} : \mathbb{W} \to \overline{\boxtimes_{\mathfrak{X}}\mathbb{W}}$.

Examples of fusion products

- $W_1 \to$, $W_2 \to$  $\to \boxtimes_{HLZ}(W_1 \otimes W_2)$ Huang-Lepowsky-Zhang
- $W \to$  $\to$ $\to$ $\boxtimes_{Li} W$ Li
-  $\mathcal{N}$ $\to$ $\to$ $\boxtimes_{\mathcal{N}} \mathbb{C}$ The coevaluation object $\int_{\mathbb{M} \in \mathrm{Mod}(\mathbb{V})} \mathbb{M} \otimes_{\mathbb{C}} \mathbb{M}'$
-  $\mathcal{T}$ $\to \boxtimes_{\mathcal{T}} \mathbb{C}$ $\simeq$ Lyubashenko construction

$$\boxtimes_{\mathrm{HLZ}}\Big(\int_{\mathbb{M} \in \mathrm{Mod}(\mathbb{V})} \mathbb{M} \otimes_{\mathbb{C}} \mathbb{M}' \Big)$$

Ends and coends

Let $\mathscr{D}$ be a category. Let $F : \mathrm{Mod}(\mathbb{V}^{\otimes N}) \times \mathrm{Mod}(\mathbb{V}^{\otimes N}) \to \mathscr{D}$ be a covariant bi-functor. Let $A \in \mathscr{D}$.

Definition

A family of morphisms $\varphi_{\mathbb{W}} : A \to F(\mathbb{W}, \mathbb{W}')$ (for all $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$) is called **dinatural** if for any $\mathbb{M} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$ and $T \in \mathrm{Hom}_{\mathbb{V}^{\otimes N}}(\mathbb{M}, \mathbb{W})$, the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\varphi_{\mathbb{W}}"] \arrow[d, "\varphi_{\mathbb{M}}"'] & F(\mathbb{W}, \mathbb{W}') \arrow[d, "F(\mathrm{id}, T^{\mathrm{t}})"] \\ F(\mathbb{M}, \mathbb{M}') \arrow[r, "F(T, \mathrm{id})"] & F(\mathbb{W}, \mathbb{M}') \end{tikzcd}$$

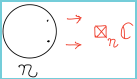
Reversing arrows defines dinatural transformation $F(\mathbb{W}', \mathbb{W}) \to A$.

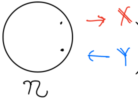
Definition

A dinatural transformation $\varphi_{\mathbb{W}}: A \to F(\mathbb{W}, \mathbb{W}')$ (for all $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$) is called an **end** if it satisfies the universal property that for any dinatural transformations $\psi_{\mathbb{W}}: B \to F(\mathbb{W}, \mathbb{W}')$ there is a unique $\Gamma \in \mathrm{Hom}_{\mathscr{D}}(B, A)$ such that $\psi_{\mathbb{W}} = \varphi_{\mathbb{W}} \circ \Gamma$ for all $\mathbb{W}$. We write $A = \int_{\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})} F(\mathbb{W}, \mathbb{W}')$.

$$\begin{tikzcd}
B \arrow[dr, "\Gamma"] \arrow[drr, bend left, "\psi_{\mathbb{W}}"] \arrow[ddr, bend right, "\psi_{\mathbb{M}}"'] & & \\
& A \arrow[r, "\varphi_{\mathbb{W}}"] \arrow[d, "\varphi_{\mathbb{M}}"'] & F(\mathbb{W}, \mathbb{W}') \arrow[d, "F(\mathrm{id}, T^{\mathrm{t}})"] \\
& F(\mathbb{M}, \mathbb{M}') \arrow[r, "F(T, \mathrm{id})"] & F(\mathbb{W}, \mathbb{M}')
\end{tikzcd}$$

Reversing arrows defines **coend** $F(\mathbb{W}', \mathbb{W}) \to A = \int^{\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})} F(\mathbb{W}', \mathbb{W})$.

The fusion product  is an end

- Since $CB\Bigg($  $\Bigg) = \mathrm{Hom}_{\mathbb{V}}(\mathbb{Y},\mathbb{X})$, $\boxtimes_{\mathfrak{N}}\mathbb{C}$ is the unique object in $\mathrm{Mod}(\mathbb{V}^{\otimes 2}) \simeq \mathrm{Mod}(\mathbb{V}) \otimes^{\mathrm{Del}} \mathrm{Mod}(\mathbb{V})$ (equivalence due to McRae 21, RHS denotes Deligne product) giving a natural equivalence
$$\mathrm{Hom}_{\mathbb{V}}(\mathbb{Y},\mathbb{X}) \simeq \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\boxtimes_{\mathfrak{N}}\mathbb{C}, \mathbb{X}\otimes_{\mathbb{C}}\mathbb{Y}')$$
- Setting $\mathbb{X}=\mathbb{Y}=\mathbb{M}\in\mathrm{Mod}(\mathbb{V})$, the identity morphism $\mathrm{id}_{\mathbb{M}}$ corresponds to a $\mathbb{V}^{\otimes 2}$-module morphism $\pi_{\mathbb{M}}:\boxtimes_{\mathfrak{N}}\mathbb{C}\to\mathbb{M}\otimes_{\mathbb{C}}\mathbb{M}'$.
- **The family $(\pi_{\mathbb{M}})_{\mathbb{M}\in\mathrm{Mod}(\mathbb{V})}$ of morphisms is dinatural. Moreover, it is an end** by a result on finite linear categories (Fuchs-Schaumann-Schweigert 16). Thus
$$\boxtimes_{\mathfrak{N}}\mathbb{C} \simeq \int_{\mathbb{M}\in\mathrm{Mod}(\mathbb{V})} \mathbb{M}\otimes_{\mathbb{C}}\mathbb{M}'$$

The sewing-factorization (SF) theorem

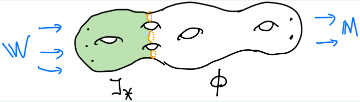
View the canonical $\gimel_{\mathfrak{X}} \in CB\Big($ 

$\Big)$ as a linear map
$\gimel^\sharp_{\mathfrak{X}} : \mathbb{W} \to \overline{\boxtimes_{\mathfrak{X}} \mathbb{W}}$.

Theorem (SF theorem, G.-Zhang arXiv:2503.23995)

We have a linear isomorphism (called the **SF isomorphism***)*

$CB\Big($ 

$\Big) \xrightarrow{\simeq} CB\Big($ 

$\Big)$

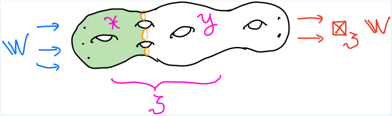
defined by $\phi^\sharp \mapsto \phi^\sharp \circ \gimel^\sharp_{\mathfrak{X}}$

- The well-definedness of this map (proved by G.-Zhang in arXiv:2411.07707) means that $\phi^\sharp \circ \gimel^\sharp_{\mathfrak{X}}$ is convergent and is also a CB.

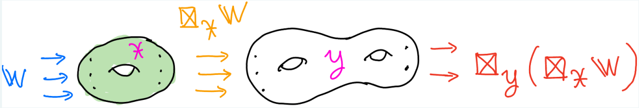
- The **transitivity of fusion products** is an easy consequence and an equivalent form of the previous SF theorem.

Theorem (G.-Zhang arXiv:2503.23995)

We have an isomorphism $\boxtimes_{\mathfrak{Z}}\mathbb{W} \simeq \boxtimes_{\mathfrak{Y}}(\boxtimes_{\mathfrak{X}}\mathbb{W})$, *pictorially*



$$\wr|$$



and, accordingly, $\daleth^\sharp_{\mathfrak{Z}} \simeq \daleth^\sharp_{\mathfrak{Y}} \circ \daleth^\sharp_{\mathfrak{X}}$

Application of the transitivity of fusion product

Recall $\boxtimes_{\mathfrak{N}}\mathbb{C} \simeq \int_{\mathbb{M}\in\mathrm{Mod}(\mathbb{V})} \mathbb{M}\otimes_{\mathbb{C}} \mathbb{M}'$.

- Applying $\boxtimes_{\mathrm{HLZ}} : \mathrm{Mod}(\mathbb{V}^{\otimes 2}) \to \mathrm{Mod}(\mathbb{V})$ to both sides of this isomorphism, we get $\boxtimes_{\mathrm{HLZ}}(\boxtimes_{\mathfrak{N}}\mathbb{C}) \simeq \mathbb{L}$ where $\mathbb{L}\in\mathrm{Mod}(\mathbb{V})$ is the **Lyubashenko construction** (Brochier-Woike 22) defined by

$$\mathbb{L} := \boxtimes_{\mathrm{HLZ}}\Big(\int_{\mathbb{M}\in\mathrm{Mod}(\mathbb{V})} \mathbb{M}\otimes_{\mathbb{C}} \mathbb{M}'\Big)$$

  - When $\mathbb{V}$ is strongly-finite and rigid, then $\mathrm{Mod}(\mathbb{V})$ is modular (McRae 21). Then $\int_{\mathbb{M}\in\mathrm{Mod}(\mathbb{V})} \mathbb{M}\otimes_{\mathbb{C}}\mathbb{M}' \simeq \int^{\mathbb{M}\in\mathrm{Mod}(\mathbb{V})} \mathbb{M}'\otimes_{\mathbb{C}}\mathbb{M}$. Since $\boxtimes_{\mathrm{HLZ}}$ is a left adjoint and hence commutes with coends, we get

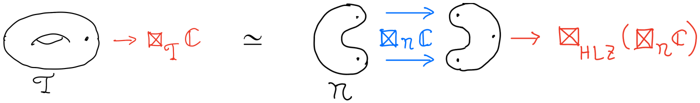
$$\mathbb{L} \simeq \int^{\mathbb{M}\in\mathrm{Mod}(\mathbb{V})} \mathbb{M}'\boxtimes_{\mathrm{HLZ}}\mathbb{M}$$

    where the RHS is the **Lyubashenko coend** (Lyubashenko 96).

Application of the transitivity of fusion product

From the previous page, we have $\boxtimes_{\mathrm{HLZ}}(\boxtimes_{\mathfrak{N}}\mathbb{C}) \simeq \mathbb{L}$.

- The transitivity of fusion products implies $\boxtimes_{\mathfrak{T}}\mathbb{C} \simeq \boxtimes_{\mathrm{HLZ}}(\boxtimes_{\mathfrak{N}}\mathbb{C})$:

  Therefore, $\boldsymbol{\boxtimes_{\mathfrak{T}}\mathbb{C} \simeq \mathbb{L}}$.

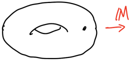
- By the definition of $\boxtimes_{\mathfrak{T}}\mathbb{C}$, for $\mathbb{M} \in \mathrm{Mod}(\mathbb{V})$ we have natural equivalence $CB($  $) \simeq \mathrm{Hom}_{\mathbb{V}}(\boxtimes_{\mathfrak{T}}\mathbb{C}, \mathbb{M})$. Therefore:

Corollary (G.-Zhang arXiv:2503.23995)

The SF isomorphism yields a natural linear isomorphism

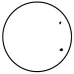
$CB($  $) \simeq \mathrm{Hom}_{\mathbb{V}}(\mathbb{L}, \mathbb{M})$

The isomorphism $CB\Big($  $\Big) \simeq \mathrm{Hom}_{\mathbb{V}}(\mathbb{L}, \mathbb{M})$

- In TFT, the torus modular functors are defined by $\mathrm{Hom}_{\mathbb{V}}(\mathbb{L}, \mathbb{M})$. The **categorical S-transform** is defined by the Hopf pairing of $\mathbb{L}$.
- Open problem (Gainutdinov-Runkel, Creutzig-Gannon, etc.): Prove that at least when $\mathbb{V}$ is strongly-finite and rigid, *the categorical S-transform agrees with the* **modular S-transform** *on* $CB\Big($  $\Big)$ *(defined by $\tau \mapsto -\frac{1}{\tau}$).*
- This conjecture is important for the construction of logarithmic full CFT, as shown by Huang-Kong in the rational case.
- The first step toward proving this conjecture must be proving $CB\Big($  $\Big) \simeq \mathrm{Hom}_{\mathbb{V}}(\mathbb{L}, \mathbb{M})$. Our work is the first one establishing such an isomorphism.

We now turn to the second part, relating (co)ends and pseudo-traces.

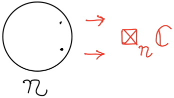
- The notion of pseudo-traces relies heavily on associative algebra structures.
- We will relate (co)ends with pseudo-traces by showing that the end $\boxtimes_{\mathfrak{N}}\mathbb{C} \simeq \int_{\mathbb{M}\in \mathrm{Mod}(\mathbb{V})} \mathbb{M} \otimes_{\mathbb{C}} \mathbb{M}'$, which is the fusion product

  $\mathfrak{N}$ $\boxtimes_{\mathfrak{N}}\mathbb{C}$, is an associative $\mathbb{C}$-algebra.

The $\mathbb{V}^{\otimes 2}$-module $\boxtimes_{\mathfrak{N}}\mathbb{C}$ as an associative algebra

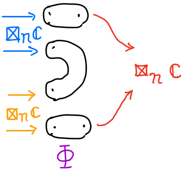
In the following, all 2-pointed spheres are standard, i.e., equivalent to $(\mathbb{P}^1; 0, \infty; z, 1/z)$.

- The canonical CB $\gimel_{\mathfrak{N}}$ of 

can be viewed as a linear $\gimel^\sharp_{\mathfrak{N}} : \mathbb{C} \to \overline{\boxtimes_{\mathfrak{N}}\mathbb{C}}$, equivalently, an element of $\overline{\boxtimes_{\mathfrak{N}}\mathbb{C}}$.
- By the SF theorem, there is a unique CB

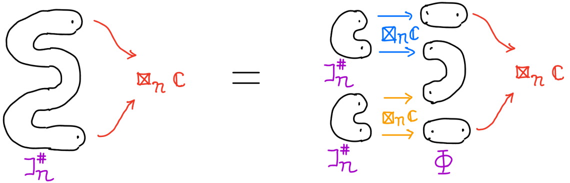
$$\Phi : (\boxtimes_{\mathfrak{N}}\mathbb{C}) \otimes (\boxtimes_{\mathfrak{N}}\mathbb{C}) \to \boxtimes_{\mathfrak{N}}\mathbb{C}$$

(algebraic closure not needed, since Φ intertwines the top and bottom actions of $L(0)$) for the following figure such that $\gimel^\sharp_{\mathfrak{N}} = \Phi(\gimel^\sharp_{\mathfrak{N}} \otimes \gimel^\sharp_{\mathfrak{N}})$.



The conformal block $\Phi : (\boxtimes_{\mathfrak{N}}\mathbb{C}) \otimes (\boxtimes_{\mathfrak{N}}\mathbb{C}) \to \boxtimes_{\mathfrak{N}}\mathbb{C}$

- Pictorially, Φ is the unique CB such that



Theorem (G.-Zhang arXiv:2508.04532)

For each $\xi, \eta \in \boxtimes_{\mathfrak{N}}\mathbb{C}$, define $\xi \star \eta = \Phi(\xi \otimes \eta)$. Then $(\boxtimes_{\mathfrak{N}}\mathbb{C}, \star)$ is a (non-unital) associative $\mathbb{C}$-algebra.

- If $\mathbb{V}$ is rational, then $\boxtimes_{\mathfrak{N}}\mathbb{C} \simeq \bigoplus_{\mathbb{M}\in\mathrm{Irr}} \mathbb{M} \otimes \mathbb{M}'$ as $\mathbb{V}^{\otimes 2}$-modules and as $\mathbb{C}$-algebras.

The functor $\mathfrak{F} : \mathrm{Mod}(\mathbb{V}) \to \mathrm{Coh}^{\mathrm{L}}(\boxtimes_{\mathfrak{N}}\mathbb{C})$

- Recall the $\mathbb{V}^{\otimes 2}$-module morphisms $\pi_{\mathbb{M}} : \boxtimes_{\mathfrak{N}}\mathbb{C} \to \mathbb{M} \otimes_{\mathbb{C}} \mathbb{M}'$ which are dinatural and make $\boxtimes_{\mathfrak{N}}\mathbb{C}$ an end.
- Let $\mathrm{Coh}^{\mathrm{L}}(\boxtimes_{\mathfrak{N}}\mathbb{C})$ be the category of finitely-generated left $\boxtimes_{\mathfrak{N}}\mathbb{C}$-modules that are quotient modules of free modules.

Theorem (G.-Zhang arXiv:2508.04532)

$\pi_{\mathbb{M}}$ is an algebra homomorphism, and $(\mathbb{M}, \pi_{\mathbb{M}})$ is an object of $\mathrm{Coh}^{\mathrm{L}}(\boxtimes_{\mathfrak{N}}\mathbb{C})$

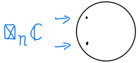
Theorem (G.-Zhang arXiv:2508.04532)

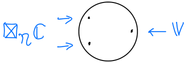
The functor $\mathfrak{F} : \mathrm{Mod}(\mathbb{V}) \xrightarrow{\simeq} \mathrm{Coh}^{\mathrm{L}}(\boxtimes_{\mathfrak{N}}\mathbb{C})$ sending $(\mathbb{M}, Y_{\mathbb{M}})$ to $(\mathbb{M}, \pi_{\mathbb{M}})$ is an isomorphism of linear categories.

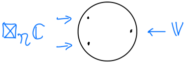
Pseudo-traces for unital finite-dimensional algebras

- Let A be a finite-dimensional unital $\mathbb{C}$-algebra and M a finite-dimensional projective left A-module.
- The projectivity is equivalence to the existence of
$$\alpha_1,\dots,\alpha_n\in\mathrm{Hom}_A(A,M)\qquad \check{\alpha}^1,\dots,\check{\alpha}^n\in\mathrm{Hom}_A(M,A)$$
satisfying $\sum_i \alpha_i\circ\check{\alpha}^i=\mathrm{id}_M$
- We have the **pseudo-trace** construction (Hattori and Stallings, 65)
$$SLF(A)\to SLF(\mathrm{End}_A(M))\qquad \phi\mapsto \mathrm{Tr}^\phi$$
$$\mathrm{Tr}^\phi(x)=\textstyle\sum_i \phi(\check{\alpha}^i\circ x\circ\alpha_i(1_A))$$
where $\boldsymbol{SLF}$=symmetric linear functionals (i.e. $\phi:A\to\mathbb{C}$ is linear and $\phi(xy)=\phi(yx)$).
- When $G\in\mathrm{Mod}^{\mathrm{L}}(A)$ is a projective generator, the pseudo-trace map $SLF(A)\to SLF(\mathrm{End}_A(G))$ is a linear isomorphism (Beliakova-Blanchet-Gainutdinov 21).

Pseudo-traces for the end $\boxtimes_{\mathfrak{N}}\mathbb{C}$

- Similarly, for each projective generator $\mathbb{G}$ of $\mathrm{Mod}(\mathbb{V}) \simeq \mathrm{Coh}^{\mathrm{L}}(\boxtimes_{\mathfrak{N}}\mathbb{C})$, we have a linear isomorphism defined by the pseudo-trace construction:
$$SLF(\boxtimes_{\mathfrak{N}}\mathbb{C}) \xrightarrow{\simeq} SLF(\mathrm{End}_{\boxtimes_{\mathfrak{N}}\mathbb{C}}(\mathbb{G})) = SLF(\mathrm{End}_{\mathbb{V}}(\mathbb{G}))$$
- $SLF(\boxtimes_{\mathfrak{N}}\mathbb{C})$ can be identified with $CB\Big($ 

$\Big)$, and hence with $CB\Big($ 

$\Big)$ by "propagation of CB".
- By the SF theorem, $CB\Big($ 

$\Big)$ is linearly isomorphic to $CB\Big($ 

$\Big)$. Therefore:

Theorem (G.-Zhang 2508.04532, conjectured by Gainutdinov-Runkel)

Let $\mathbb{G} \in \mathrm{Mod}(\mathbb{V})$ be a projective generator. Then the composition of the SF isomorphism and the pseudo-trace map yields a linear isomorphism $CB\Big($ 

$\Big) \simeq SLF(\mathrm{End}_{\mathbb{V}}(\mathbb{G}))$.

Conclusion

- Question: How are (co)ends related to pseudo-traces?
- Quick answer: The end $\int_{\mathbb{M}\in\mathrm{Mod}(\mathbb{V})} \mathbb{M}\otimes_{\mathbb{C}}\mathbb{M}'$, which is an object of $\mathrm{Mod}(\mathbb{V}^{\otimes 2})$, has a natural associative $\mathbb{C}$-algebra structure whose module category is equivalent to $\mathrm{Mod}(\mathbb{V})$, and whose SLF have a conformal-block interpretation.

  Once we have associative $\mathbb{C}$-algebras, we can run the pseudo-trace machinery.