Sewing and Factorization of Smooth and Nodal Conformal Blocks in Logarithmic CFT

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The goal of my talk

- This talk is based on the following papers.

GZ1	arXiv:2305.10180	
GZ2	arXiv:2411.07707	to appear in CCM
★ GZ3	arXiv:2503.23995	
★ Zhang 25	arXiv:2509.07720	

- The goal is to introduce **sewing-factorization (SF) theorem** in logarithmic CFT (GZ1-GZ3) and the non-equivalence of smooth and nodal conformal block functors (Zhang 25).
- Throughout my talk, I will fix a C_2-cofinite $\mathbb{N}$-graded VOA $\mathbb{V}$, which is not necessarily self dual or rational. The representation category of $\mathbb{V}$ is denoted by $\mathrm{Mod}(\mathbb{V})$.

Smooth conformal block functors

- Let $\mathfrak{X} = (C; x_1, \cdots, x_N; \eta_1, \cdots, \eta_N)$ be an N-pointed compact Riemann surface with local coordinates. The **smooth conformal block (CB) functor** is the left exact contravariant functor

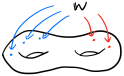
$$CB(\mathfrak{X}, -) : \mathrm{Mod}(\mathbb{V}^{\otimes N}) \to \mathcal{V}ect$$
$$\mathbb{W} \mapsto CB(\mathfrak{X}, \mathbb{W}),$$

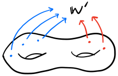
  where $CB(\mathfrak{X}, \mathbb{W})$ is the **space of smooth conformal blocks (CB)** described as follows.

- Associate $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$ to the ordered marked points $x_1, \cdots, x_N$. Then $CB(\mathfrak{X}, \mathbb{W})$ consists of linear functionals $\mathbb{W} \to \mathbb{C}$ invariant under certain intertwining properties.

Graphical calculus

- The picture for $CB(\mathfrak{X}, \mathbb{W})$ is

$CB($ 

$) = CB($ 

$)$

- The marked points is typically partitioned into several subsets.

$CB($ 

$) = CB($ 

$)$

Any CB $\phi : \mathbb{X} \otimes \mathbb{Y}' \to \mathbb{C}$ in the above space can also be viewed as a linear map $\phi^\sharp : \mathbb{X} \to \overline{\mathbb{Y}} = (\mathbb{Y}')^*$ satisfying certain intertwining properties.

Towards higher genus: sewing/composing CB

Let $\mathbb{X} \in \mathrm{Mod}(\mathbb{V}^{\otimes N}), \mathbb{Y} \in \mathrm{Mod}(\mathbb{V}^{\otimes K}), \mathbb{M} \in \mathrm{Mod}(\mathbb{V}^{\otimes L})$ and

$\phi \in CB($ 

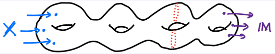
$),\quad \psi \in CB($ 

$)$

The **sewing/composition** of ϕ and ψ is defined as

$$(\psi \circ \phi)^\sharp(w) := \sum_{\lambda_\bullet \in \mathbb{C}^K} \psi^\sharp\big(P_{\lambda_\bullet}(\phi^\sharp(w))\big)$$

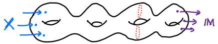
Theorem (GZ2, to appear in CCM)

$\psi \circ \phi$ *converges to a CB in* $CB($ 

$)$.

SF theorem A: horizontal composition

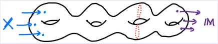
Fix $\mathbb{X} \in \mathrm{Mod}(\mathbb{V}^{\otimes N}), \mathbb{M} \in \mathrm{Mod}(\mathbb{V}^{\otimes L})$. For each $\mathbb{Y} \in \mathrm{Mod}(\mathbb{V}^{\otimes K})$, sewing CB gives a linear map

$\mathcal{S}_{\mathbb{Y}} : CB($  $) \otimes CB($  $) \to CB($  $)$

Theorem (GZ3, SF theorem A)

As $\mathbb{Y} \in \mathrm{Mod}(\mathbb{V}^{\otimes K})$ *varies, the dinatural transform* $\mathcal{S}_{\mathbb{Y}}$ *is a coend, i.e.,*

$\int^{\mathbb{Y} \in \mathrm{Mod}(\mathbb{V}^{\otimes K})} CB($  $) \otimes CB($  $)$

$\simeq CB($  $)$

Genus 0: Huang-Lepowsky-Zhang, Moriwaki.

Fusion products and canonical CB

Fix $\mathbb{X} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$ and $\mathfrak{X} =$ . Associate $\mathbb{X}$ to the blue points of $\mathfrak{X}$.

- Since every left exact linear functor from a finite $\mathbb{C}$-linear category to $\mathcal{V}ect$ is representable (Douglas-SchommerPries-Snyder 19),there exists a $\mathbb{Y}$-natural isomorphism

$$\mathrm{Hom}_{\mathbb{V}^{\otimes K}}(\boxtimes_{\mathfrak{X}}(\mathbb{X}), \mathbb{Y}) \simeq CB(\text{ } \quad \text{ })$$

  for some unique $\boxtimes_{\mathfrak{X}}(\mathbb{X}) \in \mathrm{Mod}(\mathbb{V}^{\otimes K})$ (called **fusion product**).

- The CB $\beth_{\mathfrak{X}} \in CB($  $)$ corresponding to $\mathrm{id} \in \mathrm{End}_{\mathbb{V}^{\otimes K}}(\boxtimes_{\mathfrak{X}}(\mathbb{X}))$ is called the **canonical CB**.

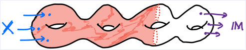
To summarize: fusion products represent smooth CB functors.

SF theorem B: fusion products

Recall the canonical CB $\gimel_{\mathfrak{X}} \in CB($  $)$.

Theorem (GZ3, SF theorem B)

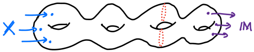
The linear map $\psi \mapsto \psi \circ \gimel_{\mathfrak{X}}$ gives an isomorphism

$CB($  $) \xrightarrow{\simeq} CB($  $)$

This isomorphism is called the ***SF isomorphism****.*

In short: replace the red part with the fusion product.

We have

$CB($  $)$

$\simeq \int^{\mathbb{Y}\in\mathrm{Mod}(\mathbb{V}^{\otimes K})} CB($  $) \otimes CB($  $)$

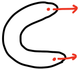
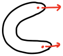
$\simeq \int^{\mathbb{Y}\in\mathrm{Mod}(\mathbb{V}^{\otimes K})} \mathrm{Hom}_{\mathbb{V}^{\otimes K}}(\boxtimes_{\mathfrak{X}}(\mathbb{X}), \mathbb{Y}) \otimes CB($  $)$

$\simeq CB($  $)$

The last isomorphism is due to Lyubashenko 96, Fuchs-Schweigert 17.

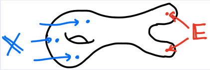
Application: self-sewing via the end $\mathbb{E}$

The end $\mathbb{E} := \int_{\mathbb{X}\in \mathrm{Mod}(\mathbb{V})} \mathbb{X} \otimes \mathbb{X}' \in \mathrm{Mod}(\mathbb{V}^{\otimes 2})$ is a fusion product of $\mathbb{C}$:

 $\boxtimes_{\mathfrak{n}} \mathbb{C} \cong \mathbb{E}$. Let $\omega \in CB($  $\boxtimes_{\mathfrak{n}} \mathbb{C} \cong \mathbb{E})$ be the canonical CB.

Corollary (GZ3)

The linear map $\psi \mapsto \omega \circ \psi$ gives an SF isomorphism

$CB($ 

$) \xrightarrow{\simeq} CB($ 

$)$

To summarize: factorization of smooth CB is given by the end $\mathbb{E}$.

Remark: When $\mathbb{V}$ is $\mathbb{N}$-graded, C_2-cofinite and rational, factorization of smooth CB is given by Damiolini-Gibney-Tarasca.

Factorization of CB in rational CFT

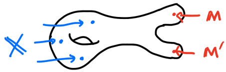
We briefy recall how algebraic geometers obtain factorization of smooth CB via nodal CB in rational CFT. We assume that $\mathbb{V}$ is $\mathbb{N}$-graded, C_2-cofinite and *rational* in the following two pages.

- Virasoro algebras, higher genus: Beilinson-Feigin-Mazur 91.
- Affine Lie algebras, higher genus: Tsuchiya-Ueno-Yamada 89, Bakalov-Kirillov 01. Looijenga 13.
- General VOA, genus 0: Nagatomo-Tsuchiya 05.
- General VOA, higher genus: Damiolini-Gibney-Tarasca 19.

I will use the setting of Damiolini-Gibney-Tarasca 19 to give an introduction.

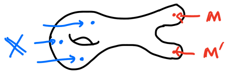
Factorization of CB in rational CFT

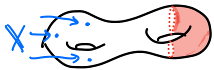
- The definition of CB can be generalized to nodal curves.
- Factorization of nodal CB is given by irreducible $\mathbb{V}$-modules.

$\bigoplus_{\mathbb{M}\in\mathrm{Irr}} CB($ 

$) \simeq CB($  $)$

- By infinitesimal smoothing of the above isomorphism, the spaces of conformal blocks form a vector bundle over $\overline{\mathcal{M}}_{g,N}$.
- In particular, we have factorization of smooth CB given by irreducible $\mathbb{V}$-modules.

$\bigoplus_{\mathbb{M}\in\mathrm{Irr}} CB($ 

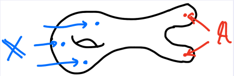
$) \simeq CB($  $)$

Factorization of nodal CB in logarithmic CFT

We return to the assumption that $\mathbb{V}$ is $\mathbb{N}$-graded and C_2-cofinite. The mode transition algebra (MTA) $\mathfrak{A}$ was introduced by Damiolini-Gibney-Krashen in 2022. As an object in $\mathrm{Mod}(\mathbb{V}^{\otimes 2})$, $\mathfrak{A}$ is defined by the two-sided induction of Zhu algebra.

Theorem (Damiolini-Gibney-Krashen 22)

We have the factorization of nodal CB:

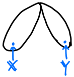
$CB($ 

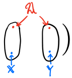
$) \simeq CB($  $)$

To summarize: factorization of nodal CB is given by the MTA $\mathfrak{A}$.

Genus 0 CB via the end and the MTA

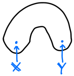
For each $\mathbb{X}, \mathbb{Y} \in \mathrm{Mod}(\mathbb{V})$, the theorem of Damiolini-Gibney-Krashen implies the factorization of genus 0 nodal CB:

$CB($ 

$) \simeq CB($ 

$) \simeq \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\mathfrak{A}, \mathbb{X}' \otimes \mathbb{Y}')$

On the other hand, by Fuchs-Schaumann-Schweigert 16, we have the factorization of genus 0 smooth CB:

$CB($ 

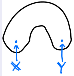
$) \simeq \mathrm{Hom}_{\mathbb{V}}(\mathbb{Y}, \mathbb{X}') \simeq \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\mathbb{E}, \mathbb{X}' \otimes \mathbb{Y}').$

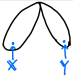
Non-equivalence of genus 0 smooth and nodal CB

In the rest of this talk, let $\mathbb{V}$ be a C_2-cofinite $\mathbb{N}$-graded VOA admitting a module that is not generated by its lowest weight subspace (e.g., the triplet algebra $\mathcal{W}_p$ and the even symplectic fermion VOA).

Theorem (Zhang 25)

There exist $\mathbb{X}, \mathbb{Y} \in \mathrm{Mod}(\mathbb{V})$ *such that*

$\dim CB($ 

$) \neq \dim CB($ 

$)$

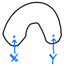
By propagation of CB, the spaces of CB associated to $\mathbb{X}, \mathbb{Y}, \mathbb{V}, \cdots, \mathbb{V}$ do not form a vector bundle on $\overline{\mathcal{M}}_{0,N}$ for $N \geqslant 4$.

The choice of $\mathbb{X}$ and $\mathbb{Y}$

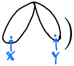
- In the proof of the above theorem, we choose $\mathbb{X}$ to be an indecomposible projective $\mathbb{V}$-module that is not generated by its lowest weight subspace, and $\mathbb{Y}$ to be an indecomposible projective module or irreducible module.
- If $\mathbb{V}$ is the triplet algebra $\mathcal{W}_p$, then $\mathbb{X}$ can be chosen to be the projective cover of X_1^-, where X_1^- is the unique irreducible module with maximal conformal weight.

The end $\mathbb{E}$ is not isomorphic to the MTA $\mathfrak{A}$

Recall that

$CB($ 

$) \simeq \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\mathbb{E}, \mathbb{X}' \otimes \mathbb{Y}')$

$CB($ 

$) \simeq \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\mathfrak{A}, \mathbb{X}' \otimes \mathbb{Y}')$

for each $\mathbb{X}, \mathbb{Y} \in \mathrm{Mod}(\mathbb{V})$. Therefore,

Corollary (Zhang 25)

The end $\mathbb{E} = \int_{\mathbb{M} \in \mathrm{Mod}(\mathbb{V})} \mathbb{M} \otimes \mathbb{M}'$ *is not isomorphic to the MTA* $\mathfrak{A}$ *as an object in* $\mathrm{Mod}(\mathbb{V}^{\otimes 2})$.