Sewing-factorization theorem and coends

Hao Zhang
Tsinghua University

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Joint work with Bin Gui

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The goal of my talk

- This talk is based on a joint project with Bin Gui.

GZ1	arXiv:2305.10180
GZ2	arXiv:2411.07707
GZ3	arXiv:2503.23995

- The main result we obtained is called **sewing-factorization (SF) theorem** for a finite logarithmic chiral CFT of arbitrary genus. The goal of my talk is to introduce SF theorem and explain why it is important to study SF theorem.
- Throughout my talk, I will fix a C_2-cofinite $\mathbb{N}$-graded VOA $\mathbb{V}$, which is not necessarily self dual or semisimple. The representation category of $\mathbb{V}$ is denoted by $\mathrm{Rep}(\mathbb{V})$.

Coends in CFT

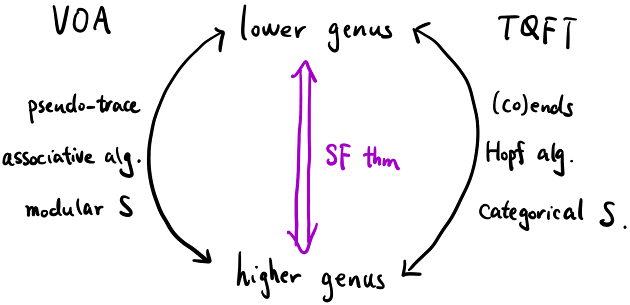
- In the literature, there are two ways to study finite logarithmic chiral CFT.

VOA community	pseudo-traces, modular S
TQFT community	(co)ends, categorical S

- The idea of "summing over all intermediate states" in physics can be realized by coend constructions in a rigorous way (Lyubashenko, Fuchs-Schweigert).
- The initial relation between pseudo-traces and coends was studied to give a formulation of non-semisimple modular Verlinde formula (Gainutdinov-Runkel). It is conjectured in their paper that "modular S=categorical S".

SF theorem and coends

SF theorem builds a bridge between pseudo-traces and (co)ends.



In this talk, I will focus on coends and describe how coends are related to SF theorem in a natural way.

(Co)ends of a bi-functor

- Let $L \in \mathbb{N}$ and $\mathcal{D}$ be a category. Choose a bi-functor $F : \mathrm{Rep}(\mathbb{V}^{\otimes L}) \times \mathrm{Rep}(\mathbb{V}^{\otimes L}) \to \mathcal{D}$ and an object $A \in \mathcal{D}$.
- A family of morphisms $\varphi_{\mathbb{W}} : F(\mathbb{W}', \mathbb{W}) \to A$ for all $\mathbb{W} \in \mathrm{Rep}(\mathbb{V}^{\otimes L})$ is called **dinatural** if for any $\mathbb{M} \in \mathrm{Rep}(\mathbb{V}^{\otimes L})$ and $T \in \mathrm{Hom}_{\mathbb{V}^{\otimes L}}(\mathbb{M}, \mathbb{W})$ (with transpose T^t), the following diagram commutes:

$$\begin{array}{ccc} F(\mathbb{W}', \mathbb{M}) & \xrightarrow{F(T^{t}, \mathrm{id}_{\mathbb{M}})} & F(\mathbb{M}', \mathbb{M}) \\ {\scriptstyle F(\mathrm{id}_{\mathbb{W}'}, T)}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_{\mathbb{M}}} \\ F(\mathbb{W}', \mathbb{W}) & \xrightarrow{\varphi_{\mathbb{W}}} & A \end{array}$$

Coends of a bi-functor

- (φ, A) is called a **coend** in $\mathcal{D}$ if it satisfies the universal property: for each $B \in \mathcal{D}$ and dinatural transformation $\psi_{\mathbb{W}} : F(\mathbb{W}', \mathbb{W}) \to B$, there is a unique $\Phi \in \mathrm{Hom}_{\mathcal{D}}(A, B)$ such that $\psi_{\mathbb{W}} = \Phi \circ \varphi_{\mathbb{W}}$ holds for all $\mathbb{W}$. If a coend exists, then it must be unique. In this case, we write

$$A = \int^{\mathbb{W} \in \mathrm{Rep}(\mathbb{V}^{\otimes L})} F(\mathbb{W}', \mathbb{W}).$$

- Reversing arrows defines ends

$$\int_{\mathbb{W} \in \mathrm{Rep}(\mathbb{V}^{\otimes L})} F(\mathbb{W}', \mathbb{W})$$

Lyubashenko's (co)ends

Assume that $\mathbb{V}$ is strongly finite and $\mathrm{Rep}(\mathbb{V})$ is rigid.

- **Lyubashenko's end** is defined by

$$\mathbb{L} = \int_{\mathbb{W}\in\mathrm{Rep}(\mathbb{V})} \mathbb{W}' \boxtimes \mathbb{W} \in \mathrm{Rep}(\mathbb{V})$$

  with dinatural transformations $\mathbb{L} \to \mathbb{W}' \boxtimes \mathbb{W}$.

- $\mathbb{L}$ is self dual and isomorphic to **Lyubashenko's coend**

$$\mathbb{L} \simeq \int^{\mathbb{W}\in\mathrm{Rep}(\mathbb{V})} \mathbb{W}' \boxtimes \mathbb{W},$$

  with dinatural transformations $\mathbb{W}' \boxtimes \mathbb{W} \to \mathbb{L}$.

The existence of Lyubashenko's (co)ends is guaranteed by rigidity.

Topological modular functors

- By Lyubashenko, Fuchs-Schweigert, the topological modular functor of a N-pointed genus g surface is described by

$$\mathrm{Hom}_{\mathbb{V}}(\mathbb{W}_1 \boxtimes \cdots \boxtimes \mathbb{W}_N \boxtimes \mathbb{L}^{\boxtimes g}, \mathbb{V}')$$

  where $\mathbb{W}_1, \cdots, \mathbb{W}_N \in \mathrm{Rep}(\mathbb{V})$, or more generally,

$$\mathrm{Hom}_{\mathbb{V}}\big(\boxtimes_{\mathrm{HLZ}} (\mathbb{W}) \boxtimes \mathbb{L}^{\boxtimes g}, \mathbb{V}' \big)$$

  where $\mathbb{W} \in \mathrm{Rep}(\mathbb{V}^{\otimes N})$ and $\boxtimes_{\mathrm{HLZ}} : \mathrm{Rep}(\mathbb{V}^{\otimes N}) \to \mathrm{Rep}(\mathbb{V})$.
- As we will show later, the space of conformal blocks is isomorphic to the topological modular functor above.

Conformal blocks

- Choose an N-pointed compact Riemann surface with local coordinates $\mathfrak{X} = (C; x_1, \cdots, x_N; \eta_1, \cdots, \eta_N)$. C *is possibly disconnected*.
- Associate $\mathbb{W} \in \mathrm{Rep}(\mathbb{V}^{\otimes N})$ to the ordered sequence $x_1, \cdots, x_N$.
- A **conformal block** (CB) is a linear map $\psi : \mathbb{W} \to \mathbb{C}$ invariant under the action of $\mathbb{V}$ and $\mathfrak{X}$ on $\mathbb{W}$ (Zhu 94, Frenkel&Ben-Zvi 04). The spaces of conformal blocks is denoted by

$$CB(\mathfrak{X}, \mathbb{W}) = CB(\quad)$$

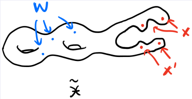
- CB functor $\mathbb{W} \in \mathrm{Rep}(\mathbb{V}^{\otimes N}) \mapsto CB(\mathfrak{X}, \mathbb{W}) \in \mathcal{V}ect$ is left exact.

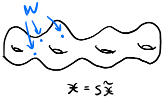
Sewing conformal blocks is dinatural

Let $\widetilde{\mathfrak{X}}$ be an $(N+2L)$-pointed surface and $\mathfrak{X} := \mathcal{S}\widetilde{\mathfrak{X}}$ be the *sewing* of $\widetilde{\mathfrak{X}}$ along L pairs of points. Here $\widetilde{\mathfrak{X}}$ is possibly disconnected.

Theorem (GZ2)

Fix $\mathbb{W} \in \mathrm{Rep}(\mathbb{V}^{\otimes N})$. For each $\mathbb{X} \in \mathrm{Rep}(\mathbb{V}^{\otimes L})$, sewing conformal blocks gives a well-defined linear map

$\mathcal{S}_{\mathbb{X}} : CB($ 

$) \to CB($ 

$)$

Moreover, $\mathcal{S}_{\mathbb{X}}$ for all $\mathbb{X} \in \mathrm{Rep}(\mathbb{V}^{\otimes L})$ is dinatural.

Miyamoto's observation

- Easy to check: any dinatural transformation into $\mathcal{V}ect$ must be surjective to be a coend.
- Unfortunately, for fixed $\mathbb{W} \in \mathrm{Rep}(\mathbb{V}^{\otimes N})$, $\mathcal{S}$ fails to be surjective in the case of self-sewing, and hence fails to be a coend. This is due to Miyamoto's observation: when $\widetilde{\mathfrak{X}}$ is a $(1 + 2 \cdot 1)$-pointed sphere and $\mathbb{W} = \mathbb{V} \in \mathrm{Rep}(\mathbb{V}^{\otimes 1})$, $\mathcal{S}$ is no longer surjective.
- *Left exact coends* and *pseudo-traces* are two methods to solve this problem.
- Pseudo-traces are studies by VOA people (Arike, Fiordalisi, Huang, Miyamoto, etc) to give a suitable formulation of modular invariance. It is a generalization of Segal's sewing, i.e., $\mathcal{S}_{\mathbb{X}}$ *when* $L = 1$.

Left exact coends, topological modular functors

In the following three pages, assume $L = 1$ for simplicity. In this case, $\mathbb{X} \in \mathrm{Rep}(\mathbb{V})$.

- Recall Lyubashenko's end $\mathbb{L}$. The dinatural transformation $\mathbb{L} \to \mathbb{X}' \boxtimes \mathbb{X}$ induces a family of morphisms

$$\mathfrak{S}_{\mathbb{X}} : \mathrm{Hom}_{\mathbb{V}}\Big(\boxtimes_{\mathrm{HLZ}} (-) \boxtimes \mathbb{X}' \boxtimes \mathbb{X} \boxtimes \mathbb{L}^{\boxtimes g-1}, \mathbb{V}'\Big) \\ \to \mathrm{Hom}_{\mathbb{V}}\Big(\boxtimes_{\mathrm{HLZ}} (-) \boxtimes \mathbb{L}^{\boxtimes g}, \mathbb{V}'\Big)$$

  in $\mathcal{L}ex(\mathrm{Rep}(\mathbb{V}^{\otimes N}), \mathcal{V}ect)$, the category of left exact contravariant functors from $\mathrm{Rep}(\mathbb{V}^{\otimes N})$ to $\mathcal{V}ect$.
- Clearly $\mathfrak{S}_{\mathbb{X}}$ is dinatural with respect to $\mathbb{X}$.

Theorem (Lyubashenko 96, Fuchs-Schweigert 17)

Assume that $\mathbb{V}$ *is strongly finite and* $\mathrm{Rep}(\mathbb{V})$ *is rigid.* $\mathfrak{S}_{\mathbb{X}}$ *is a coend into* $\mathcal{L}ex(\mathrm{Rep}(\mathbb{V}^{\otimes N}), \mathcal{V}ect)$*, i.e., it induces an equivalence*

$$\oint^{\mathbb{X}\in\mathrm{Rep}(\mathbb{V})} \mathrm{Hom}_{\mathbb{V}}\big(\boxtimes_{\mathrm{HLZ}} (-) \boxtimes \mathbb{X}' \boxtimes \mathbb{X} \boxtimes \mathbb{L}^{\boxtimes g-1}, \mathbb{V}'\big)$$

$$\simeq \mathrm{Hom}_{\mathbb{V}}\big(\boxtimes_{\mathrm{HLZ}} (-) \boxtimes \mathbb{L}^{\boxtimes g}, \mathbb{V}'\big)$$

- Although I formulate their theorem in VOA context, they actually proved in the categorical sense.
- I will show later in my talk the following equivalence

$$\mathrm{Hom}_{\mathbb{V}}\big(\boxtimes_{\mathrm{HLZ}} (-) \boxtimes \mathbb{X}' \boxtimes \mathbb{X} \boxtimes \mathbb{L}^{\boxtimes g-1}, \mathbb{V}'\big) \simeq CB(\widetilde{\mathfrak{X}}, - \otimes \mathbb{X}' \otimes \mathbb{X})$$

$$\mathrm{Hom}_{\mathbb{V}}\big(\boxtimes_{\mathrm{HLZ}} (-) \boxtimes \mathbb{L}^{\boxtimes g}, \mathbb{V}'\big) \simeq CB(\mathfrak{X}, -)$$

Left exact coends, conformal blocks

On the other hand, the sewing map gives a dinatural transformation

$$\mathcal{S}_{\mathbb{X}} : CB(\widetilde{\mathfrak{X}}, - \otimes \mathbb{X}' \otimes \mathbb{X}) \to CB(\mathfrak{X}, -)$$

in $\mathcal{L}ex(\mathrm{Rep}(\mathbb{V}^{\otimes N}), \mathcal{V}ect)$.

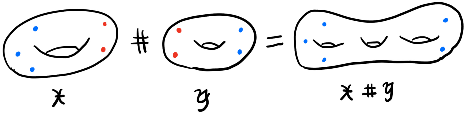
Conjecture (Gui-Z.)

Assume that $\mathbb{V}$ *is strongly finite and* $\mathrm{Rep}(\mathbb{V})$ *is rigid. Through the equivalence between* Hom *and* CB, $\mathcal{S}_{\mathbb{X}}$ *coincides with* $\mathfrak{S}_{\mathbb{X}}$. *Thus* $\mathcal{S}_{\mathbb{X}}$ *is a coend in* $\mathcal{L}ex(\mathrm{Rep}(\mathbb{V}^{\otimes N}), \mathcal{V}ect)$ *and induces*

$$\oint^{\mathbb{X} \in \mathrm{Rep}(\mathbb{V})} CB(\widetilde{\mathfrak{X}}, - \otimes \mathbb{X} \otimes \mathbb{X}') \simeq CB(\mathfrak{X}, -)$$

Disjoint sewing and coends in $\mathcal{V}ect$

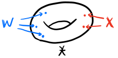
- Let $\mathfrak{X}$ be an $(N+L)$-pointed surface and $\mathfrak{Y}$ be an $(L+K)$-pointed surface. We can sew $\mathfrak{X}$ and $\mathfrak{Y}$ to get $\mathfrak{X}\#\mathfrak{Y}$, which is an $(N+K)$-pointed surface.

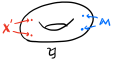


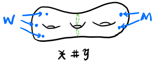
- Fix $\mathbb{W}\in \mathrm{Rep}(\mathbb{V}^{\otimes N})$, $\mathbb{M}\in \mathrm{Rep}(\mathbb{V}^{\otimes K})$. Associate $\mathbb{W},\mathbb{M},\mathbb{W}\otimes\mathbb{M}$ to the blue points of $\mathfrak{X},\mathfrak{Y},\mathfrak{X}\#\mathfrak{Y}$ respectively.

Sewing-factorization theorem

For each $\mathbb{X} \in \mathrm{Rep}(\mathbb{V}^{\otimes L})$, we already showed that sewing conformal blocks $\mathcal{S}_{\mathbb{X}}$ gives a dinatural transformation in $\mathcal{V}ect$

$\mathcal{S}_{\mathbb{X}} : CB($ 

$) \otimes CB($ 

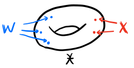
$) \to CB($ 

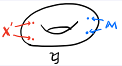
$)$

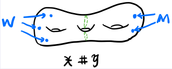
$$\psi \otimes \phi \mapsto \psi \# \phi := \mathcal{S}_{\mathbb{X}}(\psi \otimes \phi)$$

Theorem (SF theorem A, GZ3)

As $\mathbb{X} \in \mathrm{Rep}(\mathbb{V}^{\otimes L})$ varies, $\mathcal{S}_{\mathbb{X}}$ is a coend in $\mathcal{V}ect$, i.e., $\mathcal{S}$ induces

$\int^{\mathbb{X} \in \mathrm{Rep}(\mathbb{V}^{\otimes L})} CB($ 

$) \otimes CB($ 

$) \simeq CB($ 

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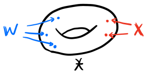
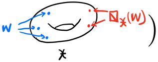
Higher genus (dual) fusion products

- In order to prove SF theorem A, we introduce higher genus (dual) fusion products to give an equivalent version of SF theorem.
- Since any left exact functor from a finite $\mathbb{C}$-linear category to $\mathcal{V}ect$ is representable, there exists $\boxbslash_{\mathfrak{X}}\mathbb{W} \in \mathrm{Rep}(\mathbb{V}^{\otimes L})$ such that we have there is an equivalence of contravariant functor

$$\mathbb{X} \mapsto \mathrm{Hom}_{\mathbb{V}^{\otimes L}}(\mathbb{X}, \boxbslash_{\mathfrak{X}}\mathbb{W}) \qquad \simeq \qquad \mathbb{X} \mapsto CB(\ \cdot\ )$$



The element corresponding to $\mathrm{id} \in \mathrm{Hom}_{\mathbb{V}^{\otimes L}}(\boxbslash_{\mathfrak{X}}\mathbb{W}, \boxbslash_{\mathfrak{X}}\mathbb{W})$ is denoted as $\omega_{\mathfrak{X}} \in CB(\ \cdot\ )$. Write $\boxtimes_{\mathfrak{X}}\mathbb{W} = (\boxbslash_{\mathfrak{X}}\mathbb{W})'$.

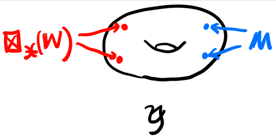


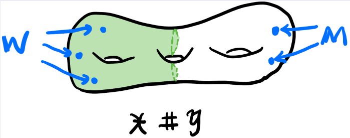
- We have an explicit construction of $\boxbslash_{\mathfrak{X}}\mathbb{W}$ as a subspace of $\mathbb{W}^*$ in GZ1. The proof of SF theorem relies heavily on this construction.

Sewing-factorization theorem

Theorem (SF theorem B, GZ3)

Sewing conformal blocks $\psi \mapsto \omega_{\mathfrak{X}} \# \psi$ *gives an isomorphism*

$CB($ 

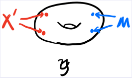
$) \xrightarrow{\simeq} CB($ 

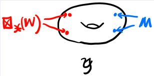
$)$

Why are two SF theorems equivalent?

Theorem (Lyubashenko 96, Fuchs-Schweigert 17)

The family of linear maps

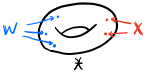
$$\mathrm{Hom}_{\mathbb{V}^{\otimes L}}\big(\mathbb{X}, \boxbslash_{\mathfrak{X}}(\mathbb{W})\big) \otimes_{\mathbb{C}} CB(\ $$ 

$$) \to CB(\ $$ 

$$)$$

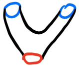
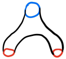
$$T \otimes \chi \mapsto \chi \circ (T^t \otimes \mathrm{id}_{\mathbb{M}})$$

for all $\mathbb{X} \in \mathrm{Rep}(\mathbb{V}^{\otimes L})$ *is a coend.*

This together with $\mathrm{Hom}_{\mathbb{V}^{\otimes L}}\big(\mathbb{X}, \boxbslash_{\mathfrak{X}}(\mathbb{W})\big) \simeq CB(\ $ 

$)$ proves the equivalence of SF theorem A and B.

Geometric realization of Lyubashenko's construction

Write $\mathfrak{P} =$  and $\mathfrak{Q} =$  .

- $\boxtimes_{\mathfrak{P}} = \boxtimes_{\mathrm{HLZ}} : \mathrm{Rep}(\mathbb{V} \otimes \mathbb{V}) \to \mathrm{Rep}(\mathbb{V})$.
- $\boxtimes_{\mathfrak{Q}}, \boxbslash_{\mathfrak{Q}} : \mathrm{Rep}(\mathbb{V}) \to \mathrm{Rep}(\mathbb{V} \otimes \mathbb{V})$.

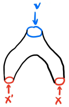
Theorem (Gui-Z. to appear)

- $\boxbslash_{\mathfrak{Q}}(\mathbb{V}) = \int^{\mathbb{X}} \mathbb{X}' \otimes \mathbb{X} \in \mathrm{Rep}(\mathbb{V} \otimes \mathbb{V})$.
- $\boxtimes_{\mathfrak{P}}(\boxbslash_{\mathfrak{Q}}(\mathbb{V})) = \int^{\mathbb{X}} \mathbb{X}' \boxtimes \mathbb{X} \in \mathrm{Rep}(\mathbb{V})$.

- This theorem implies the existence of Lyubashenko's coend in $\mathrm{Rep}(\mathbb{V})$ *without assuming rigidity.*
- If $\mathbb{V}$ is in addition rational, then $\int^{\mathbb{X}}$ can be replaced by $\bigoplus_{\mathbb{X} \in \mathrm{Irr}}$.

Dinatural transformation of coends

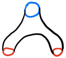
- By the definition of dual fusion products and propagation of CB, we have an isomorphism

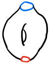
$$\mathrm{End}_{\mathbb{V}}(\mathbb{X}) \simeq CB(\ $$


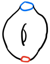
$$) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\mathbb{X}' \otimes \mathbb{X}, \boxbslash_{\mathfrak{Q}} \mathbb{V})$$

  for each $\mathbb{X} \in \mathrm{Rep}(\mathbb{V})$.
- The identity map of $\mathbb{X}$ corresponds to a morphism $\iota_{\mathbb{X}} : \mathbb{X}' \otimes \mathbb{X} \to \boxbslash_{\mathfrak{Q}} \mathbb{V}$ in $\mathrm{Rep}(\mathbb{V} \otimes \mathbb{V})$.
- Applying to functor $\boxtimes_{\mathfrak{P}} : \mathrm{Rep}(\mathbb{V} \otimes \mathbb{V}) \to \mathrm{Rep}(\mathbb{V})$, we get a morphism $\iota_{\mathbb{X}} : \mathbb{X}' \boxtimes \mathbb{X} \to \boxtimes_{\mathfrak{P}}(\boxbslash_{\mathfrak{Q}}(\mathbb{V}))$ in $\mathrm{Rep}(\mathbb{V})$.

Connection with topological modular functor

Recall that $\mathfrak{P} =$  and $\mathfrak{Q} =$  .

- SF theorem implies that $\boxtimes$  $(\mathbb{V}) \simeq \boxtimes_{\mathfrak{P}}(\boxtimes_{\mathfrak{Q}}(\mathbb{V}))$
- *From now on, assume that $\mathbb{V}$ is strongly finite and $\mathrm{Rep}(\mathbb{V})$ is rigid.* We can prove that $\boxtimes_{\mathfrak{Q}}(\mathbb{V})$ is self-dual, i.e., $\boxtimes_{\mathfrak{Q}}(\mathbb{V}) \simeq \boxbslash_{\mathfrak{Q}}(\mathbb{V})$.
- Recall that we have $\boxtimes_{\mathfrak{P}}(\boxbslash_{\mathfrak{Q}}(\mathbb{V})) = \int^{\mathbb{X}} \mathbb{X}' \boxtimes \mathbb{X}$. Therefore,

$\boxtimes$  $(\mathbb{V}) \simeq \int^{\mathbb{X} \in \mathrm{Rep}(\mathbb{V})} \mathbb{X}' \boxtimes \mathbb{X} \simeq \mathbb{L}.$

Let $\mathfrak{X}$ be an N-pointed surface with genus g and associate $\mathbb{W} \in \mathrm{Rep}(\mathbb{V}^{\otimes N})$ to the points of $\mathfrak{X}$.

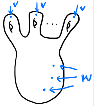
Theorem (GZ3)

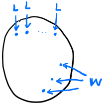
Assume that $\mathbb{V}$ is strongly finite and $\mathrm{Rep}(\mathbb{V})$ is rigid. We have an isomorphism

$$CB(\mathfrak{X}, \mathbb{W}) \simeq \mathrm{Hom}_{\mathbb{V}}\Big(\boxtimes_{\mathrm{HLZ}} (\mathbb{W}) \boxtimes \mathbb{L}^{\boxtimes g}, \mathbb{V}\Big).$$

Proof.

By SF theorem and propagation, $CB(\mathfrak{X}, \mathbb{W})$ is isomorphic to

$CB($ 

$) \simeq CB($ 

$) \simeq \mathrm{Hom}_{\mathbb{V}}\Big(\boxtimes_{\mathrm{HLZ}} (\mathbb{W}) \boxtimes \mathbb{L}^{\boxtimes g}, \mathbb{V}\Big).$

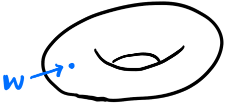
□

Torus conformal blocks

Corollary (GZ3)

Assume that $\mathbb{V}$ *is strongly finite and* $\mathrm{Rep}(\mathbb{V})$ *is rigid. Let* $\mathbb{W} \in \mathrm{Rep}(\mathbb{V})$. *We have an isomorphism*

$$CB(\quad) \simeq \mathrm{Hom}_{\mathbb{V}}(\mathbb{L}, \mathbb{W}').$$



Our result is the first that relates torus conformal blocks and $\mathbb{L}$. Before our work, no previous work on modular invariance has succeeded in establishing such a relation. This relation is crucial for relating the modular S-transform and the categorical S-transform.

Thank you for listening to my talk!