How are (co)ends related to pseudo-traces?

— Some preliminary applications of the sewing-factorization theorems

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Joint work with Hao Zhang

Goal of the talk

- This is based on my joint project with Hao Zhang

  GZ1 arXiv:2305.10180

  GZ2 arXiv:2411.07707

  ★ GZ3 arXiv:2503.23995

  GZ4 to appear this year

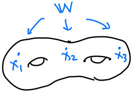
- In TQFT and VOA, modular functors/conformal blocks—particularly in genus 1—are understood in fundamentally different ways.

TQFT	ends and coends
VOA	pseudo-traces

  How to relate these two approaches? (Answer: The sewing-factorization theorem of conformal blocks proved in GZ3.)

Conformal blocks (CB)

- Throughout this talk, $\mathbb{V}$ is a C_2-cofinite $\mathbb{N}$-graded VOA.
- Fix a N-pointed compact Riemann surface with local coordinates $\mathfrak{X} = (C; x_1, \cdots, x_N; \eta_1, \cdots, \eta_N)$.
    - C is a (possibly disconnected) compact Riemann surface with distinct marked points $x_1, \ldots, x_N$. η_i is a local coordinate at x_i (i.e., an injective holomorphic function on a neighborhood of x_i sending x_i to 0).
- Associate $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$ to the *ordered* marked points $x_1, \cdots, x_N$.
- A **conformal block** (CB) is a linear map $\psi : \mathbb{W} \to \mathbb{C}$ invariant under the action defined by $\mathfrak{X}$ and $\mathbb{V}$ (Zhu 94, Frenkel&Ben-Zvi 04). The spaces of conformal blocks is denoted by $CB(\mathfrak{X}, \mathbb{W})$, or

$CB\Big($  $\Big)$

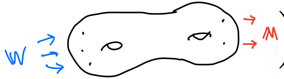
Pictorial illustration of CB

- Suppose that $\mathfrak{X}$ has two groups of marked points $x_1, \dots, x_N$ and $y_1, \dots, y_K$. We can associate $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$ to $x_1, \dots, x_N$ in order, and associate $\mathbb{M} \in \mathrm{Mod}(\mathbb{V}^{\otimes K})$ to $y_1, \dots, y_K$ in order.
  - This means associating $\mathbb{W} \otimes \mathbb{M}$ to $x_1, \dots, x_N, y_1, \dots, y_K$.
- A CB $\psi : \mathbb{W} \otimes \mathbb{M} \to \mathbb{C}$ can equivalently be viewed as a linear map $\psi^\sharp : \mathbb{W} \to \overline{\mathbb{M}'}$ satisfying certain intertwining property.

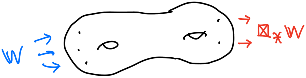
  - $\mathbb{M}'$ is the contragredient of $\mathbb{M}$, and $\overline{\mathbb{M}'}$ is the algebraic completion of $\mathbb{M}$. So $\overline{\mathbb{M}'} = \mathbb{M}^*$.

The fusion product $\boxtimes_{\mathfrak{X}}\mathbb{W}$ and the canonical CB $\daleth_{\mathfrak{X}}$

- The CB functor is left exact.
- Any left exact functor from a finite $\mathbb{C}$-linear category to $\mathcal{V}ect$ is representable (Douglas-SchommerPries-Snyder 19). So, fixing $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$, there exists $\boldsymbol{\boxtimes_{\mathfrak{X}}\mathbb{W}} \in \mathrm{Mod}(\mathbb{V}^{\otimes K})$, called the **fusion product** of $\mathbb{W}$ along $\mathfrak{X}$, yielding an equivalence of linear functors $\mathbb{M} \in \mathrm{Mod}(\mathbb{V}^{\otimes K}) \to \mathcal{V}ect$:

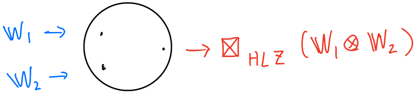
$CB\Big($ 

$\Big) \simeq \mathrm{Hom}_{\mathbb{V}^{\otimes K}}(\boxtimes_{\mathfrak{X}}\mathbb{W}, \mathbb{M})$

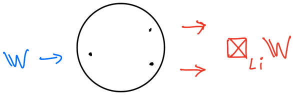
- The element $\daleth_{\mathfrak{X}} \in CB($ 

$)$ corresponding to $\mathrm{id} \in \mathrm{Hom}_{\mathbb{V}^{\otimes K}}(\boxtimes_{\mathfrak{X}}\mathbb{W}, \boxtimes_{\mathfrak{X}}\mathbb{W})$ is called the **canonical CB**.
  - $\daleth_{\mathfrak{X}}$ can be viewed as a linear map $\daleth_{\mathfrak{X}}^{\sharp} : \mathbb{W} \to \overline{\boxtimes_{\mathfrak{X}}\mathbb{W}}$.

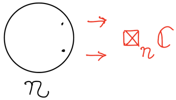
Examples of fusion products

- 

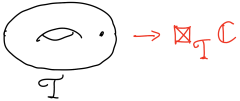
Huang-Lepowsky-Zhang

- 

Li

- 

The main subject of this talk
  - $\boxtimes_{\mathfrak{N}}\mathbb{C}$ is isomorphic to $\boxtimes_{\mathrm{Li}}\mathbb{V}$ via the "propagation of CB".

- 

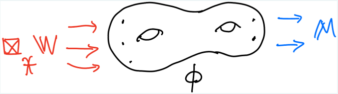
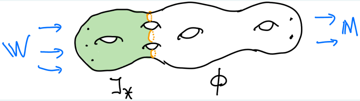
$\simeq$ Lyubashenko (GZ3, GZ4)

The sewing-factorization (SF) theorem

View the canonical $\beth_{\mathfrak{X}} \in CB($  $)$ as a linear map $\beth_{\mathfrak{X}}^{\sharp} : \mathbb{W} \to \overline{\boxtimes_{\mathfrak{X}} \mathbb{W}}$.

Theorem (SF theorem, GZ3)

We have a linear isomorphism (called the **SF isomorphism***)*

$CB($  $) \xrightarrow{\simeq} CB($  $)$

defined by $\phi^{\sharp} \mapsto \phi^{\sharp} \circ \beth_{\mathfrak{X}}^{\sharp}$

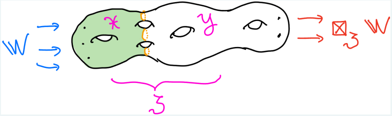
- The well-definedness of this map (proved in GZ2) means that $\phi^{\sharp} \circ \beth_{\mathfrak{X}}^{\sharp}$ is convergent and is also a CB.
  - Example: Product and iterate of intertwining operators.

Transitivity of fusion products as an SF theorem

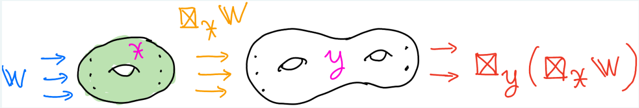
- The **transitivity of fusion products** is an easy consequence and an equivalent form of the previous SF theorem.

Theorem (Transitivity of fusion products, GZ3)

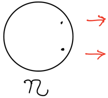
We have an isomorphism $\boxtimes_{\mathfrak{Z}}\mathbb{W} \simeq \boxtimes_{\mathfrak{Y}}(\boxtimes_{\mathfrak{X}}\mathbb{W})$, *pictorially*



$$\wr|$$

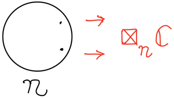


and, accordingly, $\daleth^\sharp_{\mathfrak{Z}} \simeq \daleth^\sharp_{\mathfrak{Y}} \circ \daleth^\sharp_{\mathfrak{X}}$

Next, we use the SF theorem to show that the $\mathbb{V}^{\otimes 2}$-module $\boxtimes_{\mathfrak{N}}\mathbb{C}$, which is the fusion product  $\boxtimes_{\mathfrak{N}}\mathbb{C}$, is an associative $\mathbb{C}$-algebra. We show that the category of certain left modules of $\boxtimes_{\mathfrak{N}}\mathbb{C}$ is linearly isomorphic to $\mathrm{Mod}(\mathbb{V})$.

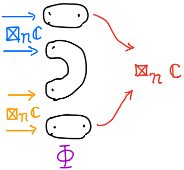
The $\mathbb{V}^{\otimes 2}$-module $\boxtimes_{\mathfrak{N}}\mathbb{C}$ as an associative algebra

In the following, all 2-pointed spheres are standard, i.e., equivalent to $(\mathbb{P}^1; 0, \infty; z, 1/z)$.

- The canonical CB $\daleth_{\mathfrak{N}}$ of  can be viewed as a linear $\daleth_{\mathfrak{N}}^{\sharp} : \mathbb{C} \to \overline{\boxtimes_{\mathfrak{N}}\mathbb{C}}$, equivalently, an element of $\overline{\boxtimes_{\mathfrak{N}}\mathbb{C}}$.
- By the SF theorem, there is a unique CB

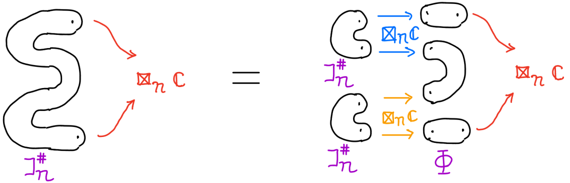
$$\Phi : (\boxtimes_{\mathfrak{N}}\mathbb{C}) \otimes (\boxtimes_{\mathfrak{N}}\mathbb{C}) \to \boxtimes_{\mathfrak{N}}\mathbb{C}$$

(algebraic closure not needed, since Φ intertwines the top and bottom actions of $L(0)$) for the following figure such that $\daleth_{\mathfrak{N}}^{\sharp} = \Phi(\daleth_{\mathfrak{N}}^{\sharp} \otimes \daleth_{\mathfrak{N}}^{\sharp})$.



The conformal block $\Phi : (\boxtimes_{\mathfrak{N}}\mathbb{C}) \otimes (\boxtimes_{\mathfrak{N}}\mathbb{C}) \to \boxtimes_{\mathfrak{N}}\mathbb{C}$

- Pictorially, Φ is the unique CB such that



Theorem (GZ4)

For each $\xi, \eta \in \boxtimes_{\mathfrak{N}}\mathbb{C}$, define $\xi \star \eta = \Phi(\xi \otimes \eta)$. Then $(\boxtimes_{\mathfrak{N}}\mathbb{C}, \star)$ is a (non-unital) associative $\mathbb{C}$-algebra which is **strongly AUF**.

- If $\mathbb{V}$ is rational, then $\boxtimes_{\mathfrak{N}}\mathbb{C} \simeq \bigoplus_{\mathbb{M}\in\mathrm{Irr}} \mathbb{M} \otimes \mathbb{M}'$ as $\mathbb{V}^{\otimes 2}$-modules and as $\mathbb{C}$-algebras.

AUF algebras

Definition

An associative $\mathbb{C}$-algebra A is called **AUF** (almost unital and finite-dimensional) if A has a family of mutually orthogonal idempotents $(p_i)_{i\in I}$ such that $A = \bigoplus_{i,j\in I} p_i A p_j$ (as vector space decomposition), and each direct summand is finite-dimensional. If A also has finitely many irreducibles, we say that A is **strongly AUF**.

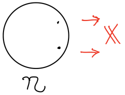
- Example: Unital and finite-dimensional algebras are strongly AUF.

Definition

Let A be AUF. A left A-module M is called **coherent** if M is finitely generated, and if M is a quotient module of $A^{\oplus J}$ for some set J. The linear category of coherent left A modules is denoted by $\mathbf{Coh^L(A)}$.

The functor $\mathfrak{F} : \mathrm{Mod}(\mathbb{V}) \to \mathrm{Coh}^{\mathrm{L}}(\boxtimes_{\mathfrak{N}}\mathbb{C})$

- For each $\mathbb{M} \in \mathrm{Mod}(\mathbb{V})$, we view $\mathbb{M} \otimes \mathbb{M}' \simeq \mathbf{End}^{\mathbf{0}}(\mathbb{M})$ as a (strongly AUF) subalgebra of $\mathrm{End}(\mathbb{M})$.
- Recall that $\boxtimes_{\mathfrak{N}}\mathbb{C} \in \mathrm{Mod}(\mathbb{V}^{\otimes 2})$ is defined by the linear equivalence

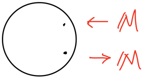
$CB\Bigg($ 

$\Bigg) \simeq \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}\big(\boxtimes_{\mathfrak{N}}\mathbb{C}, \mathbb{X}\big)$

which is natural in $\mathbb{X} \in \mathrm{Mod}(\mathbb{V}^{\otimes 2})$.

The functor $\mathfrak{F} : \mathrm{Mod}(\mathbb{V}) \to \mathrm{Coh}^{\mathrm{L}}(\boxtimes_{\mathfrak{N}}\mathbb{C})$

- In particular, setting $\mathbb{X} = \mathbb{M} \otimes \mathbb{M}' \simeq \mathrm{End}^0(\mathbb{M})$, we have

$CB\Big($  $\Big) \simeq \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}\big(\boxtimes_{\mathfrak{N}}\mathbb{C}, \mathrm{End}^0(\mathbb{M})\big)$

Theorem (GZ4)

Let $\boldsymbol{\pi}_{\mathbb{M}} : \boxtimes_{\mathfrak{N}}\mathbb{C} \to \mathrm{End}^0(\mathbb{M})$ *be the* $\mathbb{V}^{\otimes 2}$*-module morphism corresponding to* $\mathrm{id}_{\mathbb{M}}$. *Then* $\pi_{\mathbb{M}}$ *is an algebra homomorphism, and* $(\mathbb{M}, \pi_{\mathbb{M}})$ *belongs to* $\mathrm{Coh}^{\mathrm{L}}(\boxtimes_{\mathfrak{N}}\mathbb{C})$.

Theorem (GZ4)

The functor $\mathfrak{F} : \mathrm{Mod}(\mathbb{V}) \xrightarrow{\simeq} \mathrm{Coh}^{\mathrm{L}}(\boxtimes_{\mathfrak{N}}\mathbb{C})$ *sending* $(\mathbb{M}, Y_{\mathbb{M}})$ *to* $(\mathbb{M}, \pi_{\mathbb{M}})$ *is an isomorphism of linear categories.*

An elementary description of $\boxtimes_{\mathfrak{N}}\mathbb{C}$

- $\mathbb{G} \in \mathrm{Mod}(\mathbb{V})$ is called a **generator** if any $\mathbb{M} \in \mathrm{Mod}(\mathbb{V})$ is a quotient of $\mathbb{G}^{\oplus n}$ for some $n \in \mathbb{N}$. A generator which is also projective is called a **projective generator**.

Proposition (GZ4)

Let $\mathbb{G} \in \mathrm{Mod}(\mathbb{V})$ be a generator. Then $(\mathbb{G}, \pi_{\mathbb{G}})$ is faithful.

- Write $Y_{\mathbb{G}}(v,z) = \sum_{n\in\mathbb{Z}} Y_{\mathbb{G}}(v)_n z^{-n-1}$, and let $P_s : \overline{\mathbb{G}} \to \mathbb{G}(s)$ be the projection onto the generalized s-eigenspace of $L(0)$. It then follows that $\mathcal{A}_{\mathbb{G}} := \mathrm{Span}\{P_s Y_{\mathbb{G}}(v)_n P_t : s,t \in \mathbb{C}, n \in \mathbb{Z}, v \in \mathbb{V}\}$ is a subspace of $\mathrm{End}^0(\mathbb{G})$ closed under multiplication, and

$$\pi_{\mathbb{G}} : \boxtimes_{\mathfrak{N}}\mathbb{C} \xrightarrow{\simeq} \mathcal{A}_{\mathbb{G}}$$

is an algebra isomorphism. This gives an elementary characterization of $\boxtimes_{\mathfrak{N}}\mathbb{C}$.

In the rest of this talk, I give two applications of our isomorphism $\mathrm{Mod}(\mathbb{V}) \simeq \mathrm{Coh}^{\mathrm{L}}(\boxtimes_{\mathfrak{N}}\mathbb{C})$. As our first application, we show that the $\mathbb{V}^{\otimes 2}$-module $\boxtimes_{\mathfrak{N}}\mathbb{C}$ is isomorphic to the end $\int_{\mathbb{M}\in\mathrm{Mod}(\mathbb{V})} \mathbb{M} \otimes_{\mathbb{C}} \mathbb{M}'$. Using this isomorphism, we prove

$CB\Big($  $\Big) \simeq \mathrm{Hom}_{\mathbb{V}}(\mathbb{L}, \mathbb{M})$

where $\mathbb{L} \in \mathrm{Mod}(\mathbb{V})$ is the Lyubashenko construction/coend.

Ends and coends

Let $\mathscr{D}$ be a category. Let $F : \mathrm{Mod}(\mathbb{V}^{\otimes N}) \times \mathrm{Mod}(\mathbb{V}^{\otimes N}) \to \mathscr{D}$ be a covariant bi-functor. Let $A \in \mathscr{D}$.

Definition

A family of morphisms $\varphi_{\mathbb{W}} : A \to F(\mathbb{W}, \mathbb{W}')$ (for all $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$) is called **dinatural** if for any $\mathbb{M} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$ and $T \in \mathrm{Hom}_{\mathbb{V}^{\otimes N}}(\mathbb{M}, \mathbb{W})$, the following diagram commutes:

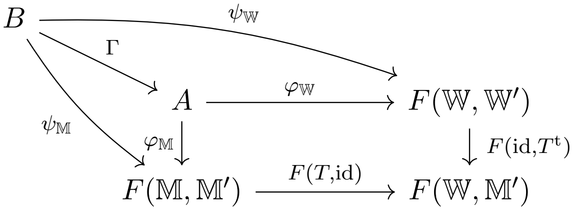
$$\begin{array}{ccc} A & \xrightarrow{\varphi_{\mathbb{W}}} & F(\mathbb{W}, \mathbb{W}') \\ {\scriptstyle \varphi_{\mathbb{M}}}\downarrow & & \downarrow{\scriptstyle F(\mathrm{id}, T^{\mathrm{t}})} \\ F(\mathbb{M}, \mathbb{M}') & \xrightarrow{F(T, \mathrm{id})} & F(\mathbb{W}, \mathbb{M}') \end{array}$$

Reversing arrows defines dinatural transformation $F(\mathbb{W}', \mathbb{W}) \to A$.

Ends and coends

Definition

A dinatural transformation $\varphi_{\mathbb{W}} : A \to F(\mathbb{W}, \mathbb{W}')$ (for all $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$) is called an **end** if it satisfies the universal property that for any dinatural transformations $\psi_{\mathbb{W}} : B \to F(\mathbb{W}, \mathbb{W}')$ there is a unique $\Gamma \in \mathrm{Hom}_{\mathscr{D}}(B, A)$ such that $\psi_{\mathbb{W}} = \varphi_{\mathbb{W}} \circ \Gamma$ for all $\mathbb{W}$. We write $A = \int_{\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})} F(\mathbb{W}, \mathbb{W}')$.



Reversing arrows defines **coend** $F(\mathbb{W}', \mathbb{W}) \to A = \int^{\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})} F(\mathbb{W}', \mathbb{W})$.

Describing $\boxtimes_{\mathfrak{N}}\mathbb{C}$ as an end

- If A is a unital finite-dimensional $\mathbb{C}$-algebra, then in $\mathrm{Bimod}(A)$,

$$A \simeq \int_{M \in \mathrm{Mod}^{\mathrm{L}}(A)} M \otimes_{\mathbb{C}} M^*$$

- A similar result holds for strongly AUF algebras. Therefore, the isomorphism $\mathrm{Mod}(\mathbb{V}) \simeq \mathrm{Coh}^{\mathrm{L}}(\boxtimes_{\mathfrak{N}}\mathbb{C})$ implies the following:

Theorem (Peter-Weyl theorem for $\boxtimes_{\mathfrak{N}}\mathbb{C}$, GZ4)

The dinatural transform $\pi_{\mathbb{M}} : \boxtimes_{\mathfrak{N}}\mathbb{C} \to \mathrm{End}^0(\mathbb{M}) = \mathbb{M} \otimes_{\mathbb{C}} \mathbb{M}'$ *(for all* $\mathbb{M} \in \mathrm{Mod}(\mathbb{V})$*) is an end in* $\mathrm{Mod}(\mathbb{V}^{\otimes 2})$*. In short,*

$$\boxtimes_{\mathfrak{N}}\mathbb{C} \simeq \int_{\mathbb{M} \in \mathrm{Mod}(\mathbb{V})} \mathbb{M} \otimes_{\mathbb{C}} \mathbb{M}'$$

Application of $\boxtimes_{\mathfrak{N}}\mathbb{C} \simeq \int_{\mathbb{M}\in\mathrm{Mod}(\mathbb{V})} \mathbb{M} \otimes_{\mathbb{C}} \mathbb{M}'$

From the previous page, we have $\boxtimes_{\mathfrak{N}}\mathbb{C} \simeq \displaystyle\int_{\mathbb{M}\in\mathrm{Mod}(\mathbb{V})} \mathbb{M} \otimes_{\mathbb{C}} \mathbb{M}'$.

- Applying $\boxtimes_{\mathrm{HLZ}} : \mathrm{Mod}(\mathbb{V}^{\otimes 2}) \to \mathrm{Mod}(\mathbb{V})$ to both sides of this isomorphism, we get $\boxtimes_{\mathrm{HLZ}}(\boxtimes_{\mathfrak{N}}\mathbb{C}) \simeq \mathbb{L}$ where $\mathbb{L} \in \mathrm{Mod}(\mathbb{V})$ is the **Lyubashenko construction** (Brochier-Woike 22) defined by

$$\mathbb{L} := \boxtimes_{\mathrm{HLZ}}\Big(\int_{\mathbb{M}\in\mathrm{Mod}(\mathbb{V})} \mathbb{M} \otimes_{\mathbb{C}} \mathbb{M}' \Big)$$

  - When $\mathbb{V}$ is strongly-finite and rigid, then $\mathrm{Mod}(\mathbb{V})$ is modular (McRae 21). Then $\int_{\mathbb{M}\in\mathrm{Mod}(\mathbb{V})} \mathbb{M} \otimes_{\mathbb{C}} \mathbb{M}' \simeq \int^{\mathbb{M}\in\mathrm{Mod}(\mathbb{V})} \mathbb{M}' \otimes_{\mathbb{C}} \mathbb{M}$. Since $\boxtimes_{\mathrm{HLZ}}$ is a left adjoint and hence commutes with coends, we get

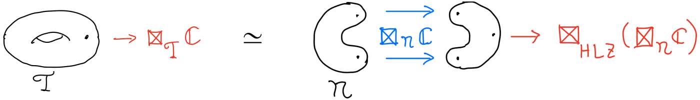
$$\mathbb{L} \simeq \int^{\mathbb{M}\in\mathrm{Mod}(\mathbb{V})} \mathbb{M}' \boxtimes_{\mathrm{HLZ}} \mathbb{M}$$

    where the RHS is the **Lyubashenko coend** (Lyubashenko 96).

Application of $\boxtimes_{\mathfrak{N}}\mathbb{C} \simeq \int_{\mathbb{M}\in\mathrm{Mod}(\mathbb{V})} \mathbb{M} \otimes_{\mathbb{C}} \mathbb{M}'$

From the previous page, we have $\boxtimes_{\mathrm{HLZ}}(\boxtimes_{\mathfrak{N}}\mathbb{C}) \simeq \mathbb{L}$.

- The transitivity of fusion products implies $\boxtimes_{\mathfrak{T}}\mathbb{C} \simeq \boxtimes_{\mathrm{HLZ}}(\boxtimes_{\mathfrak{N}}\mathbb{C})$:



  Therefore, $\boxtimes_{\mathfrak{T}}\mathbb{C} \simeq \mathbb{L}$.

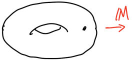
- By the definition of $\boxtimes_{\mathfrak{T}}\mathbb{C}$, for $\mathbb{M} \in \mathrm{Mod}(\mathbb{V})$ we have natural equivalence $CB($  $) \simeq \mathrm{Hom}_{\mathbb{V}}(\boxtimes_{\mathfrak{T}}\mathbb{C}, \mathbb{M})$. Therefore:

Corollary (GZ4)

The SF isomorphism yields a natural linear isomorphism

$CB($  $) \simeq \mathrm{Hom}_{\mathbb{V}}(\mathbb{L}, \mathbb{M})$

The isomorphism $CB($  $) \simeq \mathrm{Hom}_{\mathbb{V}}(\mathbb{L}, \mathbb{M})$

- In TQFT, the torus modular functors are defined by $\mathrm{Hom}_{\mathbb{V}}(\mathbb{L}, \mathbb{M})$. The **categorical S-transform** is defined by the Hopf pairing of $\mathbb{L}$.
- Open problem (Gainutdinov-Runkel, Creutzig-Gannon, etc.): Prove that at least when $\mathbb{V}$ is strongly-finite and rigid, *the categorical S-transform agrees with the* **modular S-transform** *on $CB($*  *$)$ (defined by $\tau \mapsto -\frac{1}{\tau}$).*
- This conjecture is important for the construction of logarithmic full CFT, as shown by Huang-Kong in the rational case.
- The first step toward proving this conjecture must be proving $CB($  $) \simeq \mathrm{Hom}_{\mathbb{V}}(\mathbb{L}, \mathbb{M})$. Our work is the first one establishing such an isomorphism. Before our work, no work on modular invariance has successfully related torus CB and Lyubashenko's coend $\mathbb{L}$.

As our second application of $\mathrm{Mod}(\mathbb{V}) \simeq \mathrm{Coh}^{\mathrm{L}}(\boxtimes_{\mathfrak{N}}\mathbb{C})$, we use pseudo-traces to show that $CB($  $)$ is determined by the linear category $\mathrm{Rep}(\mathbb{V})$ (i.e., the monoidal structure is irrelavant). This result, originally conjectured by Gainutdinov-Runkel (19), is much sharper than the modular invariance results of Miyamoto and Arike-Nagatomo.

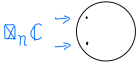
Pseudo-traces for unital finite-dimensional algebras

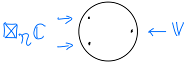
- Let A be a finite-dimensional unital $\mathbb{C}$-algebra and M a finite-dimensional projective left A-module.
- The projectivity is equivalence to the existence of
$$\alpha_1, \dots, \alpha_n \in \mathrm{Hom}_A(A, M) \qquad \breve{\alpha}^1, \dots, \breve{\alpha}^n \in \mathrm{Hom}_A(M, A)$$
satisfying $\sum_i \alpha_i \circ \breve{\alpha}^i = \mathrm{id}_M$
- We have the **pseudo-trace** construction (Hattori and Stallings, 65)
$$SLF(A) \to SLF(\mathrm{End}_A(M)) \qquad \phi \mapsto \mathrm{Tr}^\phi$$
$$\mathrm{Tr}^\phi(x) = \textstyle\sum_i \phi(\breve{\alpha}^i \circ x \circ \alpha_i(1_A))$$
where $\boldsymbol{SLF}$=symmetric linear functionals (i.e. $\phi : A \to \mathbb{C}$ is linear and $\phi(xy) = \phi(yx)$).
- When $G \in \mathrm{Mod}^{\mathrm{L}}(A)$ is a projective generator, the pseudo-trace map $SLF(A) \to SLF(\mathrm{End}_A(G))$ is a linear isomorphism (Beliakova-Blanchet-Gainutdinov 21).

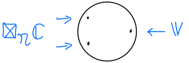
Pseudo-traces for strongly AUF algebras

- Similarly, for each projective generator $\mathbb{G}$ of $\mathrm{Mod}(\mathbb{V}) \simeq \mathrm{Coh}^{\mathrm{L}}(\boxtimes_{\mathfrak{N}}\mathbb{C})$, we have a linear isomorphism defined by the pseudo-trace construction:

$$SLF(\boxtimes_{\mathfrak{N}}\mathbb{C}) \xrightarrow{\simeq} SLF(\mathrm{End}_{\boxtimes_{\mathfrak{N}}\mathbb{C}}(\mathbb{G})) = SLF(\mathrm{End}_{\mathbb{V}}(\mathbb{G}))$$

- $SLF(\boxtimes_{\mathfrak{N}}\mathbb{C})$ can be identified with $CB($ 

$)$, and hence with $CB($ 

$)$ by "propagation of CB".
- By the SF theorem, $CB($ 

$)$ is linearly isomorphic to $CB($ 

$)$. Therefore:

Theorem (GZ4, conjectured by Gainutdinov-Runkel 19)

Let $\mathbb{G} \in \mathrm{Mod}(\mathbb{V})$ be a projective generator. Then the composition of the SF isomorphism and the pseudo-trace map yields a linear isomorphism

$CB($ 

$) \simeq SLF(\mathrm{End}_{\mathbb{V}}(\mathbb{G}))$.

The isomorphism $CB($  $) \simeq SLF(\mathrm{End}_{\mathbb{V}}(\mathbb{G}))$

- Assume that $\mathbb{V}$ is strongly-finite and rigid. Then $\mathrm{End}_{\mathbb{V}}(\mathbb{G})$ has a non-degenerate SLF, e.g. the modified trace (Gainutdinov-Runkel 20). Then $SLF(\mathrm{End}_{\mathbb{V}}(\mathbb{G}))$ is linearly isomorphic to the center $Z(\mathrm{End}_{\mathbb{V}}(\mathbb{G}))$.

Corollary

Assume that $\mathbb{V}$ is strongly-finite and rigid. Let $\mathbb{G} \in \mathrm{Mod}(\mathbb{V})$ be a projective generator. Then we have a linear isomorphism

$CB($  $) \simeq Z(\mathrm{End}_{\mathbb{V}}(\mathbb{G}))$

- Example: Let $\mathbb{V} = \mathcal{W}(p)$. Then $\mathrm{Mod}(\mathcal{W}(p)) \simeq \mathrm{Mod}^{\mathrm{L}}(\overline{U}_q(sl_2))$ as linear categories where $q = e^{\mathbf{i}\pi/p}$ (Nagatomo-Tsuchiya 09). $Z(\overline{U}_q(sl_2))$ has dimension $3p-1$ (Feigin-Gainutdinov-Semikhatov-Tipunin 06). Therefore, the space of vacuum torus CB of $\mathcal{W}(p)$ has dimension $3p-1$.

Conclusion

- Question: How are (co)ends related to pseudo-traces?
  Quick answer: The end $\int_{\mathbb{M}\in\mathrm{Mod}(\mathbb{V})} \mathbb{M}\otimes_{\mathbb{C}}\mathbb{M}'$, which is an object of $\mathrm{Mod}(\mathbb{V}^{\otimes 2})$, has a natural associative $\mathbb{C}$-algebra structure whose module category is equivalent to $\mathrm{Mod}(\mathbb{V})$.
- To summarize, our theory serves as a bridge connecting the TQFT perspective with the VOA viewpoint.

TQFT community	(co)ends, categorical S, modified traces, the Hopf algebra structure of $\mathbb{L}$,...
Our theory	the sewing-factorization of CB
VOA people	pseudo-traces, modular S, ...

The results represented in this talk are just the beginning of our story. Many more connections will be revealed in our ongoing project.