Notes on Hodge theory

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1 Hodge theory for real manifolds

1.1 Hodge $*$-operator

Let (M, g) be a n-dimensional closed oriented Riemann manifold with volume form Ω. Locally, we choose an orthonormal frame $\omega^1, \cdots, \omega^n$ with respect to g for the cotangent bundle and thus we can write $\Omega = \omega^1 \wedge \cdots \wedge \omega^n$. Denote the space of global smooth k-forms as $\Omega^k(M) := \Gamma(M, \wedge^k TM)$.

For $\omega, \eta \in \Omega^k(M)$, we write locally

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \cdots, i_k} \omega^{i_1} \wedge \cdots \wedge \omega^{i_k}$$
$$\eta = \sum_{i_1 < \cdots < i_k} \eta_{i_1, \cdots, i_k} \omega^{i_1} \wedge \cdots \wedge \omega^{i_k}.$$

Pointwisely, we define

$$\langle \omega, \eta \rangle := \sum_{i_1 < \cdots < i_k} \omega_{i_1, \cdots, i_k} \eta_{i_1, \cdots, i_k}.$$

It is easy to check $\langle \omega, \eta \rangle$ is a globally defined smooth function on M and it is independent of the choice of orthonormal frames. So it is reasonable to define

$$(\omega, \eta) := \int_M \langle \omega, \eta \rangle \, \Omega.$$

This gives an inner product on $\Omega^*(M) = \oplus_{k=0}^n \Omega^k(M)$. Note that $(-, -)$ is positive definite because Ω is nowhere vanishing.

Hodge $*$-operator $* : \Omega^k(M) \to \Omega^{n-k}(M)$ is defined by finding the 'complement' of the k-form in Ω. More precisely, if

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \cdots, i_k} \omega^{i_1} \wedge \cdots \wedge \omega^{i_k},$$

then

$$*\omega = \sum_{i_1 < \cdots < i_k} \varepsilon_{i_1, \cdots, i_k} \omega_{i_1, \cdots, i_k} \omega^1 \wedge \cdots \wedge \hat{\omega^{i_1}} \wedge \cdots \wedge \hat{\omega^{i_k}} \wedge \cdots \omega^k,$$

where

$$\varepsilon_{i_1, \cdots, i_k} = (-1)^{i_1 + \cdots + i_k + 1 + \cdots + k}.$$

Proposition 1. *Suppose $\omega, \eta \in \Omega^k(M)$. Then*

1. $*1 = \Omega$,
2. $*\Omega = 1$,
3. $* * \omega = (-1)^{k(n-k)}\omega$,
4. $\omega \wedge *\eta = \langle \omega, \eta \rangle \Omega$,
5. $\langle *\omega, *\eta \rangle = \langle \omega, \eta \rangle$.

Proof. All is straightforward from definition. We only prove 5 for example:

$$\langle *\omega, *\eta \rangle \Omega = *\omega \wedge * * \eta = (-1)^{k(n-k)} * \omega \wedge \eta = \eta \wedge *\omega = \langle \eta, \omega \rangle \Omega.$$

Then 5 follows from the fact: Ω is nowhere vanishing. □

Recall $d : \Omega^{k-1}(M) \to \Omega^k(M)$. Define $\delta = (-1)^{n(k-1)+1} * d* : \Omega^k(M) \to \Omega^{k-1}(M)$.

Proposition 2. *δ is the adjoint operator of d with respect to $(-,-)$, i.e.,*

$$(d\omega, \eta) = (\omega, \delta\eta),$$

for $\omega \in \Omega^{k-1}(M), \eta \in \Omega^k(M)$.

Proof. By direct computation,

$$\begin{aligned} d(\omega \wedge *\eta) &= d\omega \wedge *\eta + (-1)^{k-1}\omega \wedge d * \eta \\ &= d\omega \wedge *\eta + (-1)^{n(k-1)}\omega \wedge * * d * \eta \\ &= d\omega^*\eta - \omega \wedge *\delta\eta. \end{aligned}$$

By Stoke's theorem,

$$(d\omega, \eta) = \int_M d\omega^*\eta = \int_M \omega \wedge *\delta\eta = (\omega, \delta\eta).$$

□

1.2 Harmonic forms and Hodge decomposition

Definition 1. *$\Delta := d\delta + \delta d : \Omega^k(M) \to \Omega^k(M)$ is called* ***Hodge-Laplace operator****. If $\omega \in \Omega^k(M)$ satisfies $\Delta\omega = 0$, then ω is called a* ***harmonic form****.*

Proposition 3. *Hodge-Laplace operator satisfies:*

1. *Δ is self-adjoint, i.e., $(\Delta\omega, \eta) = (\omega, \Delta\eta)$ for all differtial forms.*
2. *Δ is positive, i.e., $(\Delta\omega, \omega) \geq 0$ and the equality holds if and only if $\Delta\omega = 0$.*
3. $*\Delta = \Delta*$.

Proof. To show Δ is self-adjoint, it suffices to assume ω and η are both k-forms. Then

$$\begin{aligned} (\Delta\omega, \eta) &= (d\delta\omega, \eta) + (\delta d\omega, \eta) \\ &= (\delta\omega, \delta\eta) + (d\omega, d\eta) \\ &= (\omega, \Delta\eta). \end{aligned}$$

Note that the above identity gives

$$(\Delta\omega, \omega) = (\delta\omega, \delta\omega) + (d\omega, d\omega) \geq 0,$$

and $(\Delta\omega, \omega) = 0$ if and only if $\delta\omega = 0$ and $d\omega = 0$, if and only if $\Delta\omega = 0$.

To show Δ commutes with $*$, assume ω is a k-form, then

$$*\delta\omega = (-1)^{n(k-1)+1} * *d * \omega = (-1)^k d * \omega.$$

Similarly, $\delta * \omega = (-1)^{k+1} * d\omega$. So

$$*d\delta\omega = (-1)^k \delta * \delta\omega = \delta d * \omega.$$

Similarly, $*\delta d = d\delta *$. Thus,

$$*\Delta = *d\delta + *\delta d = \delta d * + d\delta * = \Delta *$$

□

As we can see in the proof, ω is a harmonic form if and only if $d\omega = 0$ and $\delta\omega = 0$.

Denote $\mathcal{H}^k(M)$ as the vector space of harmonic k-forms on M. We will see $\mathcal{H}^k(M)$ is actually isomorphic to the de Rham cohomology of M.

Proposition 4. *Suppose $\omega \in \mathcal{H}^k(M)$.*

1. *ω has minimal norm in the de Rham cohomology class $[\omega]$. More precisely, for any $(k-1)$-form η, $(\omega + d\eta, \omega + d\eta) \geq (\omega, \omega)$, and the equality holds if and only if $d\eta = 0$.*

2. *$*\omega \in \mathcal{H}^{n-k}(M)$.*

Proof. Suppose η is a $(k-1)$-form. Then

$$\begin{aligned}(\omega + d\eta, \omega + d\eta) &= (\omega, \omega) + 2(\omega, d\eta) + (d\eta, d\eta) \\ &= (\omega, \omega) + 2(\delta\omega, \eta) + (d\eta, d\eta) \\ &= (\omega, \omega) + (d\eta, d\eta) \geq (\omega, \omega).\end{aligned}$$

The equality holds if and only if $d\eta = 0$. $*\omega \in \mathcal{H}^{n-k}(M)$ follows immediately from $\Delta * = *\Delta$. □

Theorem 1 (Poincaré duality)**.** *Hodge $*$-operator gives an isomorphism $\mathcal{H}^k(M) \cong \mathcal{H}^{n-k}(M)$.*

Proof. $* : \mathcal{H}^k(M) \to \mathcal{H}^{n-k}(M)$ is an isomorphism because $** = (-1)^{k(n-k)}$. □

To relate $\mathcal{H}^k(M)$ with the usual de Rham cohomology, we introduce our main result in the real case.

Theorem 2 (Hodge decomposition)**.** *There exists an isomorphism of vector spaces:*

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus d\Omega^{k-1}(M) \oplus \delta\Omega^{k+1}(M).$$

More precisely, for any k-form ω, there exists a unique decomposition

$$\omega = \omega_h + d\sigma + \delta\tau,$$

where $\omega_h \in \mathcal{H}^k(M), \sigma \in \Omega^{k-1}(M), \tau \in \Omega^{k+1}(M)$. When ω is closed, the decomposition reduces to

$$\omega = \omega_h + d\sigma.$$

Theorem 3 (Poincaré duality for de Rham cohomology)**.** *We have an isomorphism $\mathcal{H}^k(M) \cong H^k_{dR}(M)$, which gives Poincaré duality for de Rham cohomology*

$$H^k_{dR}(M) \cong H^{n-k}_{dR}(M).$$

Proof. Define a linear map $\iota : \mathcal{H}^k(M) \to H^k_{dR}(M)$ by $\omega \mapsto [\omega]$. ι is injective by Proposition 4. ι is surjective by Hodge decomposition. So ι is an isomorphism. □

2 Hodge theory for complex manifolds

2.1 Dolbeault cohomology

Let X be a n-dimensional complex manifold. Denote Ω_X^p as the sheaf of holomorphic p-forms on X and $\mathcal{A}_{X,\mathbb{C}}^k$ as the sheaf of complex k-forms on X. Recall the decomposition of sheaves

$$\mathcal{A}_{X,\mathbb{C}}^k = \bigoplus_{p+q=k} \mathcal{A}_X^{p,q},$$

and the differential

$$\partial : \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p+1,q}$$
$$\bar{\partial} : \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p,q+1},$$

where $\mathcal{A}_X^{p,q}$ is the sheaf of forms of type (p,q) on X. Recall these sheaves $\mathcal{A}^{p,q}$ are acyclic, i.e., have trivial higher cohomology from partition of unity. Then Dolbeault cohomology with respect to differential forms is defined by

$$H^{p,q}(X) := H^q(\mathcal{A}^{p,\bullet}(X), \bar{\partial}) = \frac{\operatorname{Ker}(\bar{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X))}{\operatorname{Im}(\bar{\partial} : \mathcal{A}^{p,q-1}(X) \to \mathcal{A}^{p,q}(X))}.$$

In fact, the Dolbeault cohomology is isomorphic to sheaf cohomology of Ω_X^p, i.e.,

$$H^{p,q}(X) \cong H^q(X, \Omega_X^p).$$

To see this, it suffices to see the acyclic resolution

$$0 \to \Omega_X^p \to \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \to \cdots$$

from $\bar{\partial}$-Poincaré lemma.

Let E be a complex vector bundle over X and $\mathcal{A}^{p,q}(E)$ denote the sheaf defined by

$$U \mapsto \mathcal{A}^{p,q}(U, E) := \Gamma(U, \wedge^{p,q} X \otimes E),$$

where the tensor product is taken over $\mathcal{O}_X$. Locally, a section α of $\mathcal{A}^{p,q}(E)$ can be written as $\alpha = \sigma\alpha_i \otimes s_i$ with α and s_i local sections of $\mathcal{A}_X^{p,q}$ and E respectively.

Lemma 1. *Suppose E is a holomorphic vector bundle. There exists a natural $\mathbb{C}$-linear operator $\bar{\partial}_E : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E)$ with $\bar{\partial}_E^2 = 0$ and which satisfies the Leibniz rule*

$$\bar{\partial}_E(f \cdot \alpha) = \bar{\partial}(f) \wedge \alpha + f\bar{\partial}_E(\alpha).$$

Proof. Choose a local trivialization $s = (s_1, \cdots, s_r)$ of E and write $\alpha \in \mathcal{A}^{p,q}(E)$ locally as $\alpha = \sum \alpha_i \otimes s_i$, where $\alpha \in \mathcal{A}_X^{p,q}$. Define

$$\bar{\partial}_E \alpha := \sum \bar{\partial}(\alpha_i) \otimes s_i.$$

Suppose we choose another holomorphic trivialization $s' = (s'_1, \cdots, s'_r)$ and obtain an operator $\bar{\partial}'_E$. Let $s_i = \sum_j \psi_{ij} s'_j$, where ψ_{ij} is the holomorphic transition function. Then

$$\begin{aligned}
\bar{\partial}'_E \alpha &= \bar{\partial}'_E\Big(\sum \alpha_i \otimes \sum_j \psi_{ij} s'_j\Big) \\
&= \sum_{i,j} \bar{\partial}(\alpha_i \psi_{ij}) \otimes s'_j \\
&= \sum_{i,j} \bar{\partial}(\alpha_i)\psi_{ij} \otimes s'_j = \bar{\partial}_E(\alpha).
\end{aligned}$$

So $\bar{\partial}_E = \bar{\partial}'_E$ is independent of the choice of local trivialization. Therefore, $\bar{\partial}_E^2 = 0$ since $\bar{\partial}^2 = 0$. From Leibniz rule of $\bar{\partial}$,

$$\begin{aligned}
\bar{\partial}_E(f \cdot \alpha) &= \bar{\partial}_E\big(\sum f\alpha_i \otimes s_i\big) \\
&= \sum \bar{\partial}(f\alpha_i) \otimes s_i \\
&= \sum \big(\bar{\partial}(f) \wedge \alpha_i + f\bar{\partial}(\alpha_i)\big) \otimes s_i \\
&= \bar{\partial}(f) \wedge \alpha + f\bar{\partial}_E(\alpha).
\end{aligned}$$

□

The above lemma gives a complex $(\mathcal{A}^{p,\bullet}(X,E),\bar{\partial}_E)$, whose cohomology is called Dolbeault cohomology of the holomorphic vector bundle E:

$$H^{p,q}(X,E) := H^q(\mathcal{A}^{p,\bullet}(X,E),\bar{\partial}_E) = \frac{\mathrm{Ker}(\bar{\partial}_E : \mathcal{A}^{p,q}(X,E) \to \mathcal{A}^{p,q+1}(X,E))}{\mathrm{Im}(\bar{\partial}_E : \mathcal{A}^{p,q-1}(X,E) \to \mathcal{A}^{p,q}(X,E))}.$$

Similarly, Dolbeault cohomology of holomorphic vector bundles is isomorphic to sheaf cohomology:

$$H^{p,q}(X,E) \cong H^q(X, E \otimes \Omega_X^p),$$

which follows from acyclic resolution of $E \otimes \Omega_X^p$:

$$0 \to E \otimes \Omega_X^p \to \mathcal{A}^{p,0}(E) \xrightarrow{\bar{\partial}_E} \mathcal{A}^{p,1}(E) \to \cdots$$

To summarize what we obtain:

Dolbeault cohomology of a holomorphic vector bundle E = sheaf cohomology of $E \otimes \Omega_X^p$.

2.2 Hermitian and Kähler structure on complex manifolds

We briefly recall some definition in this subsection.

Let X be a complex manifold with almost complex structure I and complex dimension n. A Riemann metric g on X is an hermitian structure if for any point $x \in X$, the scalar product g_x is compatible with I, i.e.,

$$g_x(Iv, Iw) = g_x(v,w).$$

The induced real $(1,1)$-form $\omega := g(I(),())$ is called the fundamental form of hermitian manifold (X,g). After complexification, the fundamental form ω is locally of the form

$$\omega = \frac{i}{2}\sum_{i,j=1}^{n} h_{ij} dz_i \wedge d\bar{z}_j,$$

where $(h_{ij}(x))$ is a positive definition matrix for each $x \in X$. It is not difficult to see that the hermitian structure is uniquely determined by I and ω. The hermitian structure g is called a Kähler structure if ω is closed. Denote the hermitian extension of g by $g_{\mathbb{C}}$.

The Hodge $*-$operator $* : \bigwedge_{\mathbb{C}}^k X \to \bigwedge_{\mathbb{C}}^{2n-k} X$ is similar to the one defined for Riemann manifolds (X,g) with the natural volume form Ω. More precisely, $*$ is defined by $\alpha \wedge *\bar{\beta} = g_{\mathbb{C}}(\alpha,\beta)\Omega$. When restricted to $\bigwedge_{\mathbb{C}}^k X$, $*$ reduces to the usual Hodge $*-$operator for Riemann manifolds.

With Hodge $*-$operator, we can define several adjoint operators. Regard (X,g) as a real Riemann manifold with natural volume form Ω. Since X has even dimension, the adjoint operator $d^* = \delta$ is exactly $d^* = - * \circ d \circ *$. Analogously, one defines ∂^* and $\bar{\partial}^*$ as $\partial^* := - * \circ \bar{\partial} \circ *$ and $\bar{\partial}^* = - * \circ \partial \circ *$ to make $d^* = \partial^* + \bar{\partial}^*$ valid.

Therefore, it is natural to define the Laplacian operator associated to $d, \partial, \bar{\partial}$: $\Delta := d^*d + dd^*$, $\Delta_\partial = \partial^*\partial + \partial\partial^*$, $\Delta_{\bar{\partial}} = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$. Note that $* : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{n-q,n-p}(X)$, $\partial^* : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p-1,q}(X)$, $\bar{\partial}^* : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q-1}(X)$ and $\Delta_\partial, \Delta_{\bar{\partial}}$ perserve bidegrees.

2.3 Hodge decomposition for Kähler manifolds

Syppose X is a complex manifold with an hermitian structure g and natural fundamental form Ω. The hermitian extension of g is denoted by $g_{\mathbb{C}}$. Define an hermitian product on $\mathcal{A}_{\mathbb{C}}^*(X)$ by

$$(\alpha,\beta) := \int_X g_{\mathbb{C}}(\alpha,\beta) * 1 = \int_X g_{\mathbb{C}}(\alpha,\beta)\Omega.$$

Note that the value of g on $\mathcal{A}^*(X)$ is the exactly the inner product defined in the first section.

With respect to the hermitian product $(-,-)$. The degree decomposition

$$\mathcal{A}_{\mathbb{C}}^*(X) = \bigoplus_k \mathcal{A}_{\mathbb{C}}^k(X)$$

and the bidegree decomposition

$$\mathcal{A}_{\mathbb{C}}^k(X) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X)$$

are both orthogonal decompositions and each component $\mathcal{A}^{p,q}(X)$ is an infinite dimensional normed vector space with scalar product $(-,-)$ and the induced norm $\|\alpha\|^2 = (\alpha,\alpha)$.

Proposition 5. *Suppose X is a closed hermitian manifold. Then with respect to $(-,-)$, the operators $d^*, \partial^*, \bar{\partial}^*$ are actually adjoint operators of $d, \partial, \bar{\partial}$.*

Proof. We give a proof for ∂ for example. For $\alpha \in \mathcal{A}^{p-1,q}(X)$ and $\beta \in \mathcal{A}^{p,q}(X)$,

$$\begin{aligned}(\partial\alpha, \beta) &= \int_X g_{\mathbb{C}}(\partial\alpha, \beta) * 1 = \int_X \partial\alpha \wedge *\bar{\beta} \\ &= \int_X \partial(\alpha \wedge *\bar{\beta}) - (-1)^{p+q-1} \int_X \alpha \wedge \partial(*\bar{\beta})\end{aligned}$$

Since $\alpha \wedge *\bar{\beta}$ is of bidegree $(n-1, n)$, $\partial(\alpha \wedge *\bar{\beta}) = d(\alpha \wedge *\bar{\beta})$. So

$$\int_X \partial(\alpha \wedge *\bar{\beta}) = 0$$

by Stokes' theorem. Using $*^2 = (-1)^k$ on $\mathcal{A}^k(X)$, we compute

$$\int_X \alpha \wedge \partial(*\bar{\beta}) = (-1)^{2n-(p+q)+1} \int_X g_{\mathbb{C}}(\alpha, -\partial^*\beta) * 1 = (-1)^{2n-(p+q)}(\alpha, \partial^*\beta).$$

So $(\partial\alpha, \beta) = (\alpha, \partial^*\beta)$. □

In the case of real manifolds, we interpret de Rham cohomology by harmonic forms. In the complex case, we will apply similar approach. For the differential d, the spaces of harmonic k-forms and (p,q)-forms (which are defined similarly in the real case) are denoted by $\mathcal{H}^k(X,g)$ and $\mathcal{H}^{p,q}(X,g)$. For ∂ and $\bar{\partial}$, we have analogous definition.

Definition 2. *A k-form is called $\bar{\partial}$-harmonic if $\Delta_{\bar{\partial}}\alpha = 0$ and define the spaces of $\bar{\partial}$-harmonic k-forms and (p,q)-forms by $\mathcal{H}^k_{\bar{\partial}}(X,g)$ and $\mathcal{H}^{p,q}_{\bar{\partial}}(X,g)$. ∂-harmonic forms are analogous.*

Proposition 6. *Suppose (X,g) is a closed hermitian manifold. A form α is $\bar{\partial}$-harmonic (resp. ∂-harmonic) if and only if $\bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$ (resp. $\partial\alpha = \partial^*\alpha = 0$).*

Proof. The $\bar{\partial}$ case follows from the identity

$$\begin{aligned}(\Delta_{\bar{\partial}}\alpha, \alpha) &= (\bar{\partial}^*\bar{\partial}\alpha + \bar{\partial}\bar{\partial}^*\alpha, \alpha) \\ &= \|\bar{\partial}^*(\alpha)\|^2 + \|\bar{\partial}\|^2.\end{aligned}$$

□

Proposition 7. *Suppose (X,g) is an hermitian manifold. Then*

1. *$\mathcal{H}^k_{\bar{\partial}}(X,g) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar{\partial}}(X,g)$ and $\mathcal{H}^k_{\partial}(X,g) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\partial}(X,g)$.*
2. *If (X,g) is Kähler, then both decompositions coincide with $\mathcal{H}^k(X,g)_{\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X,g)$. In particular, $\mathcal{H}^k(X,g)_{\mathbb{C}} = \mathcal{H}^k_{\bar{\partial}}(X,g) = \mathcal{H}^k_{\partial}(X,g)$.*

Proof. Suppose $\alpha = \sum \alpha^{p,q}$ is the bidegree decomposition of a $\bar{\partial}$-harmonic form α. Then

$$0 = \sum \Delta_{\bar{\partial}}(\alpha^{p,q})$$

is also a bidegree decomposition, which implies $\Delta_{\bar{\partial}}(\alpha^{p,q}) = 0$ for all p,q. The proof of ∂ decomposition is analogous.

The second assertion follows from the identity $\Delta_{\partial} = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta$ on a Kähler manifold. □

Theorem 4 (Serre duality for harmonic forms)**.** *Suppose (X,g) is a compact connected hermitian manifold. Then the pairing*

$$\mathcal{H}^{p,q}_{\bar{\partial}}(X,g) \times \mathcal{H}^{n-p,n-q}_{\bar{\partial}}(X,g) \to \mathbb{C}, \quad (\alpha,\beta) \mapsto \int_X \alpha \wedge \beta$$

is non-degenerate. This yields an isomorphism

$$\mathcal{H}^{p,q}_{\bar{\partial}}(X,g) \cong \mathcal{H}^{n-p,n-q}_{\bar{\partial}}(X,g)^*.$$

Proof. Suppose $0 \neq \alpha \in \mathcal{H}^{p,q}_{\bar\partial}(X, g)$. Then

$$\int_X \alpha \wedge *\bar\alpha = \|\alpha\|^2 > 0$$

implies the pairing is non-degenerate. □

Our main result is:

Theorem 5 (Hodge decomposition of harmonic forms)**.** *Let (X, g) be a compact hermitian manifold. Then there exists two natural orthogonal decompositions*

$$\mathcal{A}^{p,q}(X) = \partial\mathcal{A}^{p-1,q}(X) \oplus \mathcal{H}^{p,q}_{\partial}(X, g) \oplus \partial^*\mathcal{A}^{p+1,q}(X),$$
$$\mathcal{A}^{p,q}(X) = \bar\partial\mathcal{A}^{p-1,q}(X) \oplus \mathcal{H}^{p,q}_{\bar\partial}(X, g) \oplus \bar\partial^*\mathcal{A}^{p+1,q}(X).$$

Moreover, $\mathcal{H}^{p,q}(X, g)_{\mathbb{C}}$ are all finite dimensional and if X is a Kähler manifold, then $\mathcal{H}^{p,q}(X, g)_{\mathbb{C}} = \mathcal{H}^{p,q}_{\partial}(X, g) = \mathcal{H}^{p,q}_{\bar\partial}(X, g)$.

The most nontrivial part is the existence of decomposition. We will not prove it in our note. The significance of Kähler condition is that we can forget all about '$\bar\partial$ or $\bar\partial$-harmonic', and replace them by 'harmonic'.

Corollary 1. *Suppose (X, g) is a compact hermitian manifold. Then the canonical map $\mathcal{H}^{p,q}_{\bar\partial}(X, g) \to H^{p,q}(X)$ is an isomorphism.*

Proof. The canonical map $\mathcal{H}^{p,q}_{\bar\partial}(X, g) \to H^{p,q}(X)$ is given by $\alpha \mapsto [\alpha]$, where $[\alpha]$ is a cohomology class of α. This map is injective because any harmonic coboundary $\bar\partial\beta$ must satisfy $\bar\partial^*\bar\partial\beta = 0$. But $0 = (\bar\partial^*\bar\partial\beta, \beta) = (\bar\partial\beta, \bar\partial\beta) = \|\bar\partial\beta\|^2 = 0$ implies $\bar\partial\beta = 0$. To show this map is surjective, it suffices to show any $\bar\partial-$closed $(p, q)-$form β must be cohomological to a $\bar\partial-$harmonic form. By Hodge decomposition, write $\beta = \bar\partial\beta_1 + \beta_h + \bar\partial^*\beta_2$, where β_h is harmonic. Since $\bar\partial\beta = 0$, it follows $\bar\partial\bar\partial^*\beta_2 = 0$, which implies $\bar\partial^*\beta_2$ by a similar argument in the injective case. □

The proof shows the Hodge decomposition for closed forms does not contain the terms $\partial^*\mathcal{A}^{p+1,q}(X)$ or $\bar\partial^*\mathcal{A}^{p+1,q}(X)$, just as the case in real manifolds.

Corollary 2. *Let (X, g) be a compact Kähler manifold. Then there exists a decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

which does not depend on the Kähler structure.

Proof. Since X is Kähler,

$$H^k(X, \mathbb{C}) = \mathcal{H}^k(X, g)_{\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X, g) = \bigoplus_{p+q=k} H^{p,q}(X).$$

To show the decomposition is independent of the Kähler structure, we choose two Kähler metric g, g' and $\alpha \in \mathcal{H}^{p,q}(X, g), \alpha' \in \mathcal{H}^{p,q}(X, g')$, which induce same elements in $H^{p,q}(X)$. So α and α' differ by some $\bar\partial\gamma$, i.e., $\alpha' = \alpha + \bar\partial\gamma$. Then $d\bar\partial\gamma = 0$. By Hodge decomposition for d,

$$\bar\partial\gamma = d\beta + \beta_h.$$

But $0 = (\gamma, \bar\partial^*\beta_h) = (\bar\partial\gamma, \beta_h) = (\beta_h, \beta_h)$ implies $\beta_h = 0$. So $\bar\partial\gamma \in d(\mathcal{A}^{k-1}_{\mathbb{C}}(X))$ and α, α' induces the same de Rham cohomology class in $H^k(X, \mathbb{C})$. □

3 Serre duality

In this section, we will give a generalized version of Hodge decomposition and Serre duality on a holomorphic vector bundle. Serre duality, together with Riemann-Roch theorem and Kodaira vanishing theorem, is significant in controlling the cohomology of holomorphic vector bundles. Most parts of this section are routine.

3.1 Hermitian structure on vector bundles

Suppose M is a real manifold and $E \to M$ is a complex vector bundle.

Definition 3. *An hermitian structure h on $E \to M$ is an hermitian scalar product h_x on each fiber $E(x)$ which depends differtiably on x, The pair (E, h) is called an hermitian vector bundle.*

If $\psi : E|_U \cong U \times \mathbb{C}^r$ is a trivialization over some open subset U, then h_x is given by a positive-definite hermitian matrix $(h_{ij}(x))$ for each $x \in U$. The definition says $(h_{ij}(x))$ relies differtiably on $x \in U$.

Example 1. *If (X, g) is an hermitian manifold, then the tangent, cotangent and form bundles have natural hermitian structures. Moreover, if (E, h) is an hermitian vector bundle over (X, g), then $\bigwedge^{p,q} X \otimes E$ have natural hermitian structures.*

Proposition 8. *Every complex vector bundle admits an hermitian structure.*

Proof. Choose an open covering $X = \bigcup U_i$ trivializing E and glue the constant hermitian structure on the trivial bundles $U_i \times \mathbb{C}^r$ by means of partition of unity. The resulting product is hermitian because positive linear combination of hermitian product is again hermitian. □

Now let (X, g) be an hermitian manifold of complex dimension n and (E, h) is an hermitian vector bundle. Denote the induced hermitian structure on $\bigwedge^{p,q} X \otimes E$ by $(-, -)$. We may interpret h as a $\mathbb{C}-$antilinear isomorphism $h : E \cong E^*$.

Definition 4. *Hodge $*$-operator $\bar{*}_E : \bigwedge^{p,q} X \otimes E \to \bigwedge^{n-p,n-q} X \otimes E^*$ is defined by*

$$\bar{*}_E(\varphi \otimes s) = \bar{*}_E(\varphi) \otimes h(s) = \overline{*(\varphi)} \otimes h(s) = *(\bar{\varphi}) \otimes h(s).$$

Hodge $*$-operator $\bar{*}_E$ is a $\mathbb{C}$-antilinear isomorphism that depends on g and h. Similarly, we check easily

$$(\alpha, \beta) * 1 = \alpha \wedge \bar{*}_E(\beta)$$

where $\wedge$ means taking usual wedge products in the form part and evaluation map in the bundle part. Moreover, $\bar{*}_E \circ \bar{*}_E = (-1)^{p+q}$ on $\bigwedge^{p,q} X \otimes E$. From now on, denote $\mathcal{A}^{p,q}(E)$ by sheaf of sections of $\bigwedge^{p,q} X \otimes E$. We will not distinguish $\mathcal{A}^{p,q}(E)$ and $\bigwedge^{p,q} X \otimes E$ from now on.

Define $\bar{\partial}_E : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E)$ by $\bar{\partial}_E(\alpha \otimes s) = \bar{\partial}(\alpha) \otimes s$ and its adjoint operator $\bar{\partial}_E^* : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q-1}(E)$ by $\bar{\partial}_E^* = -\bar{*}_{E^*} \circ \bar{\partial}_{E^*} \circ \bar{*}_E$. The Laplacian operator $\Delta_E : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q}(E)$ is defined by $\Delta_E = \bar{\partial}_E^* \bar{\partial}_E + \bar{\partial}_E \bar{\partial}_E^*$.

Definition 5. *A section $\alpha \in \mathcal{A}^{p,q}(E)$ is called harmonic if $\Delta_E(\alpha) = 0$. The space of harmonic forms is denoted by $\mathcal{H}^{p,q}(X, E)$.*

Since $\bar{*}_E$ commutes with Δ_E, $\bar{*}_E$ restricts to a $\mathbb{C}-$antilinear isomorphism $\bar{*}_E : \mathcal{H}^{p,q}(X, E) \to \mathcal{H}^{p,q}(X, E^*)$.

From now on, we suppose (X, g) is a compact hermitian manifold. Define a hermitian scalar product on $\mathcal{A}^{p,q}(X, E)$ by

$$(\alpha, \beta) = \int_X (\alpha, \beta) * 1,$$

where $(-, -)$ inside the integral is the pointwise hermitian inner product on $\mathcal{A}^{p,q}(X, E)$.

Proposition 9. *1. $\bar{\partial}_E^*$ is actually the adjoint operator of $\bar{\partial}_E$ and Δ_E is self-adjoint with respect to $(-, -)$.*

2. $\alpha \in \mathcal{A}^{p,q}(X, E)$ is harmonic if and only if $\bar{\partial}_E(\alpha) = \bar{\partial}_E^*(\alpha) = 0$.

Proof. Analogous to Proposition 2 and 3. □

3.2 Serre duality on vector bundles

To give Serre duality on vector bundles, we have to give a generalized version of Hodge decomposition for vector bundles.

Theorem 6 (Hodge decomposition for vector bundles)**.** *Suppose (X, g) is a compact hermitian manifold and (E, h) is an hermitian vector bundle. We have Hodge decomposition*

$$\mathcal{A}^{p,q}(X, E) = \bar{\partial}_E \mathcal{A}^{p,q-1}(X, E) \oplus \mathcal{H}^{p,q}(X, E) \oplus \bar{\partial}_E^* \mathcal{A}^{p,q+1}(X, E)$$

and $\mathcal{H}^{p,q}(X, E)$ is finite dimensional for each p and q.

If we choose $E = \mathcal{O}_X$, the above theorem reduces to the usual Hodge decomposition.

Corollary 3. *The natural map $\mathcal{H}^{p,q}(X,E) \to H^{p,q}(X,E)$ is an isomorphism. In particular, Dolbeault cohomology $H^{p,q}(X,E) \cong H^q(X, E \otimes \Omega_X^p)$ is finite dimensional.*

Proof. To show the map is injective, choose a harmonic coboundary $\bar{\partial}_E \beta$. So

$$0 = (\bar{\partial}_E^* \bar{\partial}_E \beta, \beta) = (\bar{\partial}_E \beta, \bar{\partial}_E \beta) = \|\bar{\partial}_E \beta\|^2$$

implies $\bar{\partial}_E \beta = 0$.

To show the map is surjective, choose a closed element $\alpha \in \mathcal{A}^{p,q}(X,E)$ and by Hodge decomposition,

$$\alpha = \bar{\partial}_E \alpha_1 + \alpha_h + \bar{\partial}_E^* \alpha_2.$$

Since $\bar{\partial}_E \alpha = 0$, $\bar{\partial}_E \bar{\partial}_E^* \alpha_2 = 0$ and then $\bar{\partial}_E^* \alpha_2 = 0$. So $\alpha = \bar{\partial}_E \alpha_1 + \alpha_h$, i.e., the image of $\alpha_h \in \mathcal{H}^{p,q}(X,E)$ under the natural map is the cohomology class $[\alpha]$. □

By this corollary, we see any (nonzero) Dolbeault cohomology class can be represented by a (nonzero) harmonic element.

Theorem 7 (Serre duality for vector bundles)**.** *Suppose X is a compact complex manifold. For any holomorphic vector bundle E on X, define a pairing*

$$H^{p,q}(X,E) \times H^{n-p,n-q}(X,E^*) \to \mathbb{C}$$

by

$$(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta.$$

Then the pairing is non-degenerate.

Proof. This pairing is well-defined by Stokes' theorem. Choose hermitian structures h and g on E and X. For any nonzero cohomology class in $H^{p,q}(X,E)$, we can find a nonzero harmonic element $\alpha \in \mathcal{H}^{p,q}(X,E)$ representing the given class. Define $\beta = \bar{*}_E \alpha \in \mathcal{H}^{n-p,n-q}(X,E^*)$. We check

$$\int_X \alpha \wedge \beta = \int_X \alpha \wedge \bar{*}_E \alpha = \int_X (\alpha, \alpha) * 1 = \|\alpha\|^2 \neq 0.$$

So this pairing is non-degenerate. □

Corollary 4. *For any holomorphic vector bundle E over a compact complex manifold X, there exist $\mathbb{C}-$linear isomorphisms:*

$$H^q(X, E \otimes \Omega^p) \cong H^{p,q}(X,E) \cong H^{n-p,n-q}(X,E^*)^* \cong H^{n-q}(X, E^* \otimes \Omega^{n-p})^*$$